Calculus Essentials Ebook

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Precalculus

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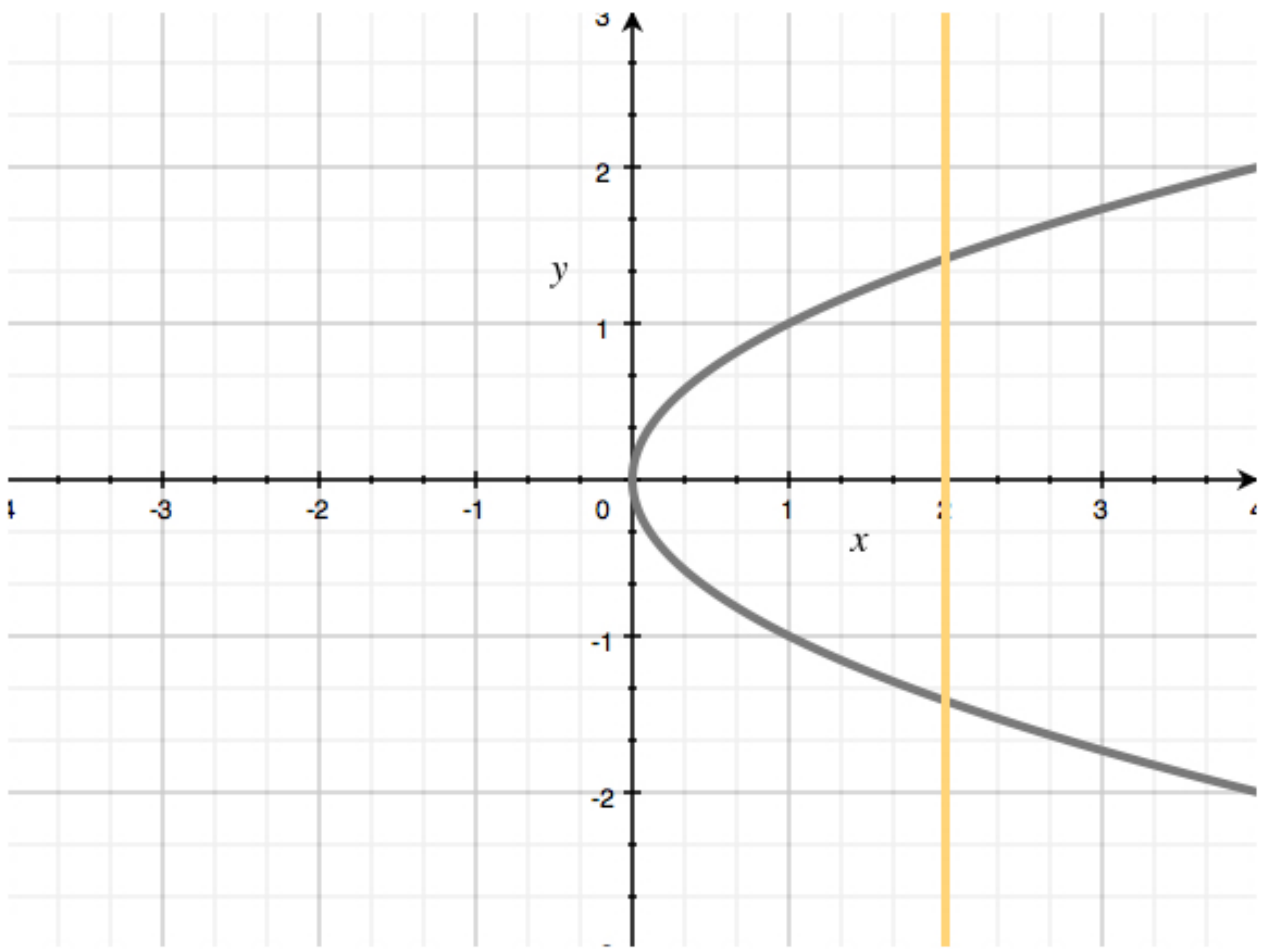
Most of the equations you'll encounter in calculus are functions. Since not all equations are functions, it's important to understand that only functions can pass the Vertical Line Test.

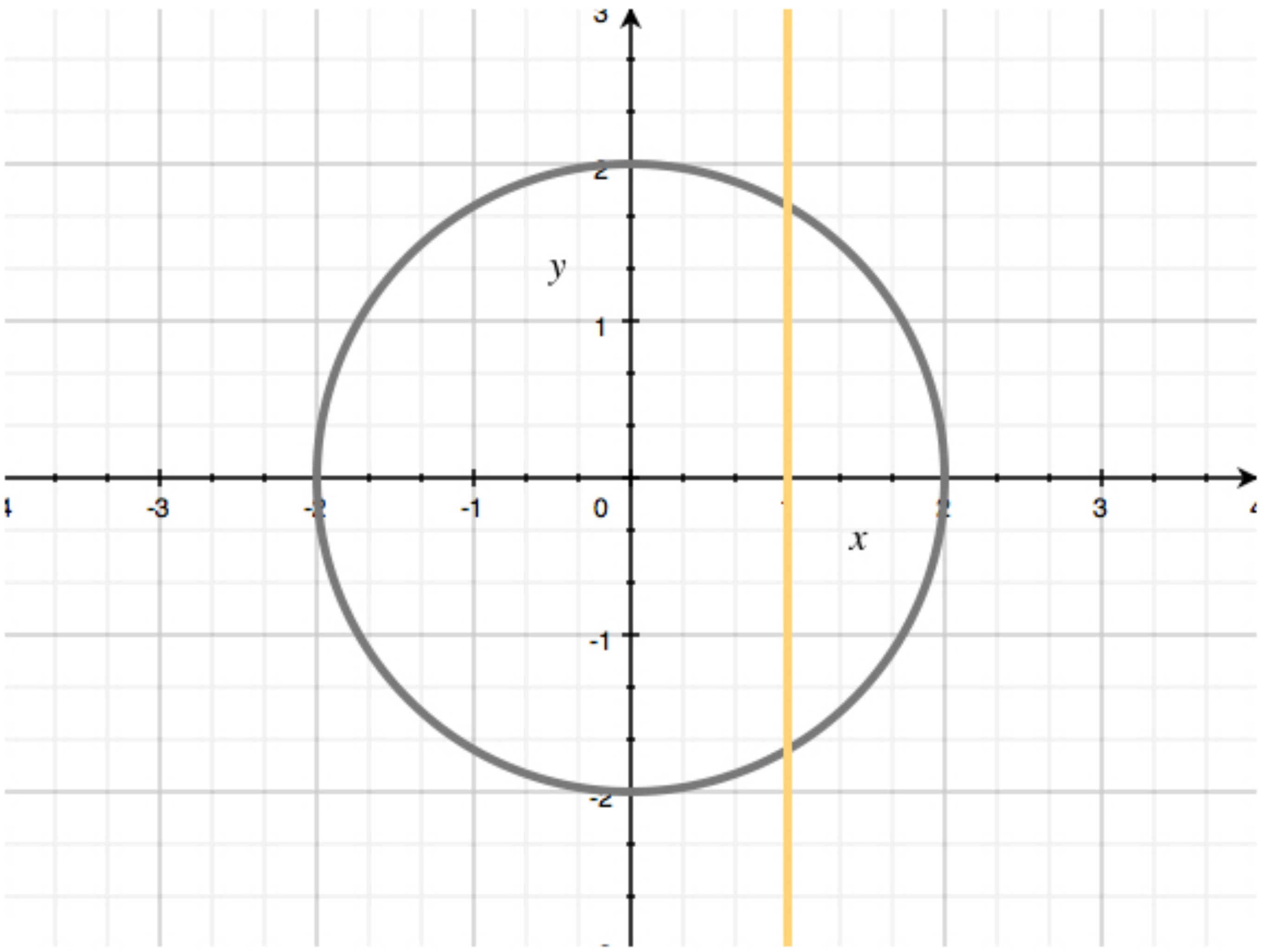
Vertical line test

Most of the equations you’ll encounter in calculus are functions. Since not all equations are functions, it’s important to understand that only functions can pass the Vertical Line Test. In other words, in order for a graph to be a function, no completely vertical line can cross its graph more than once. The graph below doesn't pass the Vertical Line Test because a vertical line intersects it more than once.



Passing the Vertical Line Test also implies that the graph has only one output value for y for any input value of x. You know that an equation is not a function if y can be two different values at a single x value.

You know that the circle below is not a function because any vertical line you draw between $x = -2$ and $x = 2$ will cross the graph twice, which causes the graph to fail the Vertical Line Test. In fact, circles can never be called functions because they'll never pass the Vertical Line Test.



You can also test this algebraically by plugging in a point between -2 and 2 for x, such as $x=1$.

Example

Determine algebraically whether or not $x^2+y^2=1$ is a function.

Plug in 0 for x and simplify.

$$(0)^2+y^2=1$$

$$y^2=1$$

$$\sqrt{y^2} = \sqrt{1}$$

$$y = \pm 1$$

Looking at it another way, at $x = 0$, y can be both 2 and -2 . Since a function can only have one unique output value for y for any input value of x, the function fails the Vertical Line Test and is therefore not a function. We've now proven with both the graph and with algebra that this circle is not a function.

Limits & Continuity 2

The limit of a function is the value the function approaches at a given value of x, regardless of whether the function actually reaches that value.

Idea of the limit

The limit of a function is the value the function approaches at a given value of x, regardless of whether the function actually reaches that value.

For an easy example, consider the function

$$f(x) = x + 1$$

When $x = 5$, $f(x) = 6$. Therefore, 6 is the limit of the function at $x = 5$, because 6 is the value that the function approaches as the value of x gets closer and closer to 5.

I know it's strange to talk about the value that a function "approaches." Think about it this way: If you set $x = 4.9999$ in the function above, then $f(x) = 5.9999$. Similarly, if you set $x = 5.0001$, then $f(x) = 6.0001$.

You can begin to see that as you get closer to $x = 5$, whether you're approaching it from the 4.9999 side or the 5.0001 side, the value of $f(x)$ gets closer and closer to 6.

x	4.9999	5.0000	5.0001
$f(x)$	5.9999	6.0000	6.0001

In this simple example, the limit of the function is clearly 6 because that is the actual value of the function at that point; the point is defined. However, finding limits gets a little trickier when we start dealing with points of the graph that are undefined.

In the next section, we'll talk about when limits do and do not exist, and some more creative methods for finding the limit.

One-sided limits

General vs. one-sided limits

When you hear your professor talking about limits, he or she is usually talking about the general limit. Unless a right- or left-hand limit is specifically specified, you're dealing with a general limit.

The general limit exists at the point $x = c$ if

1. The left-hand limit exists at $x = c$,
2. The right-hand limit exists at $x = c$, and
3. The left- and right-hand limits are equal.

These are the three conditions that must be met in order for the general limit to exist. The general limit will look something like this:

$$\lim_{x \to 2} f(x) = 4$$

You would read this general limit formula as "The limit of f of x as x approaches 2 equals 4."

Left- and right-hand limits may exist even when the general limit does not. If the graph approaches two separate values at the point $x = c$ as you approach c from the left- and right-hand side of the graph, then separate left- and right-hand limits may exist.

Left-hand limits are written as

$$\lim_{x \to 2^-} f(x) = 4$$

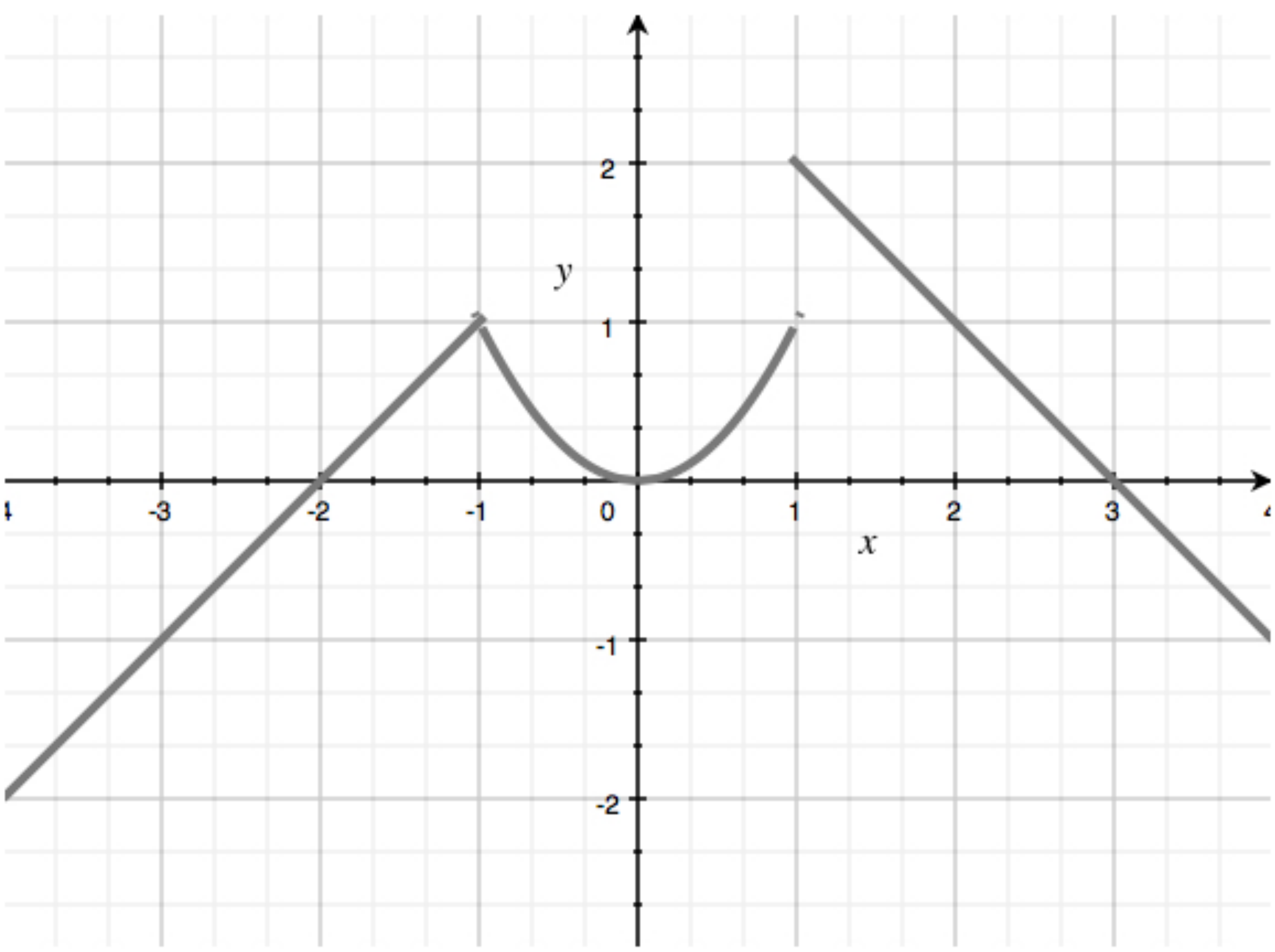
The negative sign after the 2 indicates that we're talking about the limit as we approach 2 from the negative, or left-hand side of the graph.

Right-hand limits are written as

$$\lim_{x \to 2^+} f(x) = 4$$

The positive sign after the 2 indicates that we're talking about the limit as we approach 2 from the positive, or right-hand side of the graph.

In the graph on the right, the general limit exists at $x = -1$ because the left- and right- hand limits both approach 1. On the other hand, the general limit does not exist at $x = 1$ because the left-hand and right-hand limits are not equal, due to a break in the graph.



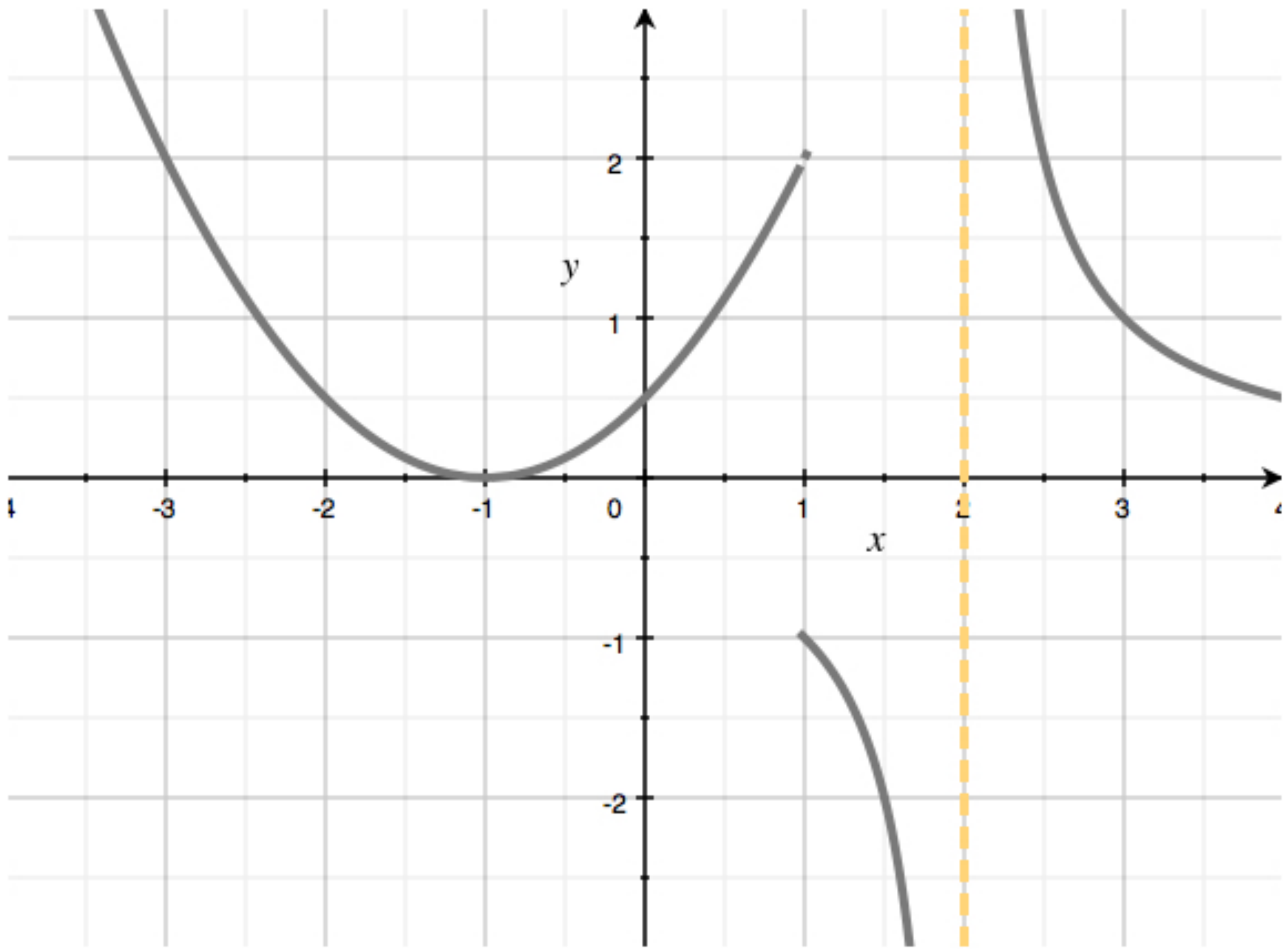
You can see from the graph that the left- and right-hand limits are equal at $x = -1$, but not at $x = 1$.

Where limits don't exist

We already know that a general limit does not exist where the left- and right-hand limits are not equal. Limits also do not exist whenever we encounter a vertical asymptote.

There is no limit at a vertical asymptote because the graph of a function must approach one fixed numerical value at the point $x = c$ for the limit to exist at c. The graph at a vertical asymptote is increasing and/or decreasing without bound, which means that it is approaching infinity instead of a fixed numerical value.

In the graph below, separate right- and left-hand limits exist at $x = 1$, so the general limit does not exist at that point. The left-hand limit is 2, because that is the value that the graph approaches as you trace the graph from left to right. On the other hand, the right-hand limit is -1, since that's the value that the graph approaches as you trace the graph from right to left.



Where there is a vertical asymptote at $x = 2$, the left-hand limit is $-\infty$, and the right-hand limit is $+\infty$. However, the general limit does not exist at the vertical asymptote because the left- and right-hand limits are unequal. So we can say that the general limit does not exist at $x = 1$ or at $x = 2$.

Solving limits with substitution

Sometimes you can find the limit just by plugging in the number that your function is approaching. You could have done this with our original limit example, $f(x) = x + 1$. If you just plug 5 into this function, you get 6, which is the limit of the function. Below is another example, where you can simply plug in the number you're approaching to solve for the limit.

Example

Evaluate the limit.

$$\lim_{x \to -2} x^2 + 2x + 6$$

Plug in -2 for x and simplify.

$$(-2)^2 + 2(-2) + 6$$

$$4 - 4 + 6$$

$$6$$

Solving limits with factoring

When you can't just plug in the value you're evaluating, your next approach should be factoring.

Example

Evaluate the limit.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

Just plugging in 4 would give us a nasty $0/0$ result. Therefore, we'll try factoring instead.

$$\lim_{x \to 4} \frac{(x + 4)(x - 4)}{x - 4}$$

Canceling $(x - 4)$ from the top and bottom of the fraction leaves us with something that is much easier to evaluate:

$$\lim_{x \to 4} x + 4$$

Now the problem is simple enough that we can use substitution to find the limit.

$$4 + 4$$

$$8$$

Solving limits with conjugate method

This method can only be used when either the numerator or denominator contains exactly two terms. Needless to say, its usefulness is limited. Here's an example of a great, and common candidate for the conjugate method.

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}$$

In this example, the substitution method would result in a 0 in the denominator. We also can't factor and cancel anything out of the fraction. Luckily, we have the conjugate method. Notice that the numerator has exactly two terms, $\sqrt{4+h}$ and -2.

Conjugate method to the rescue! In order to use it, we have to multiply by the conjugate of whichever part of the fraction contains the radical. In this case, that's the numerator. The conjugate of two terms is those same two terms with the opposite sign in between them.

Notice that we multiply both the numerator and denominator by the conjugate, because that's like multiplying by 1, which is useful to us but still doesn't change the value of the original function.

Example

Evaluate the limit.

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}$$

Multiply the numerator and denominator by the conjugate.

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} \cdot \left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$$

Simplify and cancel the h.

$$\lim_{h \to 0} \frac{(4+h)+2\sqrt{4+h}-2\sqrt{4+h}-4}{h(\sqrt{4+h}+2)}$$

$$\lim_{h \to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)}$$

$$\lim_{h \to 0} \frac{h}{h(\sqrt{4+h}+2)}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{4+h}+2}$$

Since we're evaluating at 0, plug that in for h and solve.

$$\frac{1}{\sqrt{4+0}+2}$$

$$\frac{1}{2+2}$$

$$\frac{1}{4}$$

Remember, if you’re trying to evaluate a limit and substitution, factoring, and conjugate method all don’t work, you can always go back to the simple method of plugging in a number very close to the value you’re approaching and solve for the limit that way.

Continuity

You should have some intuition about what it means for a graph to be continuous. Basically, a function is continuous if there are no holes, breaks, jumps, fractures, broken bones, etc. in its graph.

You can also think about it this way: A function is continuous if you can draw the entire thing without picking up your pencil. Let's take some time to classify the most common types of discontinuity.

Point discontinuity

Point discontinuity exists when there is a hole in the graph at one point. You usually find this kind of discontinuity when your graph is a fraction like this:

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

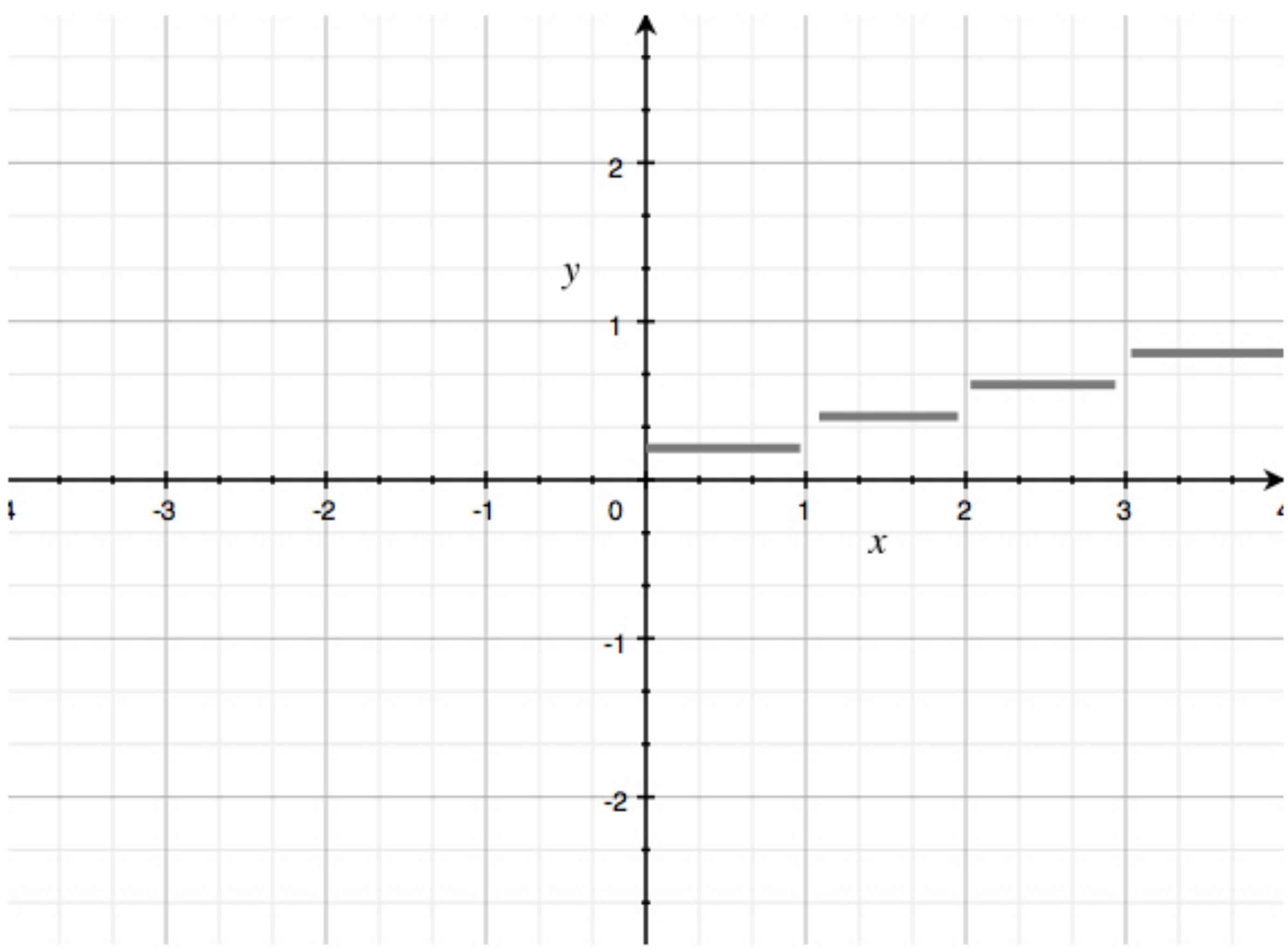
In this case, the point discontinuity exists at $x = -4$, where the denominator would equal 0. This function is defined and continuous everywhere, except at $x = -4$. The graph of a point discontinuity is easy to pick out because it looks totally normal everywhere, except for a hole at a single point.

Jump discontinuity

You’ll usually encounter jump discontinuities with piecewise-defined functions, which is a function for which different parts of the domain are defined by different functions. A common example used to illustrate piecewise-defined functions is the cost of postage at the post office.

Below is an example of how the cost of postage might be defined as a function, as well as the graph of the cost function. They tell us that the cost per ounce of any package lighter than 1 pound is 20 cents per ounce; that the cost of every ounce from 1 pound to anything less than 2 pounds is 40 cents per ounce; etc.

$$f(x) = \begin{cases} 0.2 & 0 < x < 1 \\ 0.4 & 1 \le x < 2 \\ 0.6 & 2 \le x < 3 \\ 0.8 & 3 \le x < 4 \\ 1.00 & 4 \le x \end{cases}$$

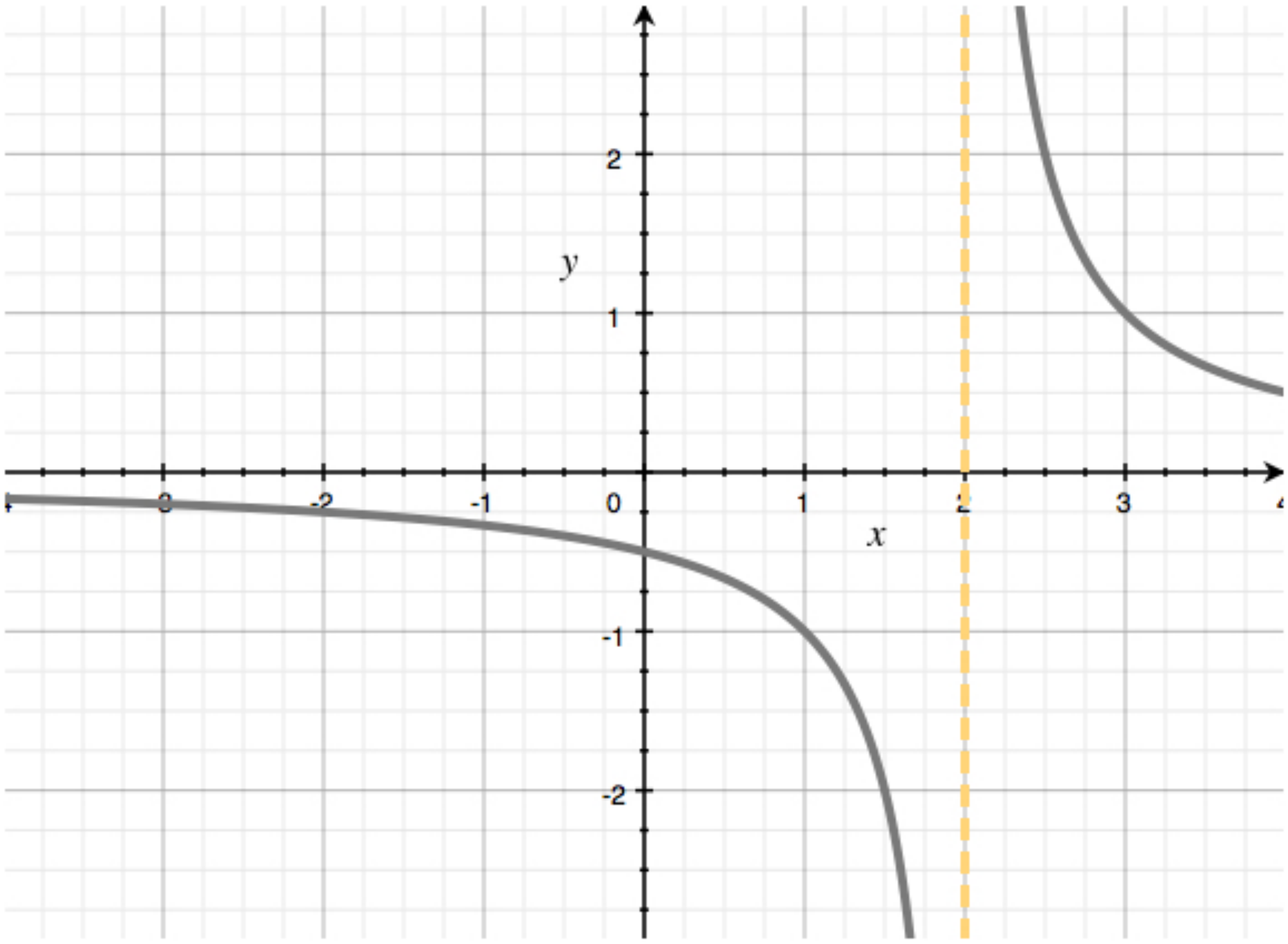


Every break in this graph is a point of jump discontinuity. You can remember this by imagining yourself walking along on top of the first segment of the graph. In order to continue, you'd have to jump up to the second segment.

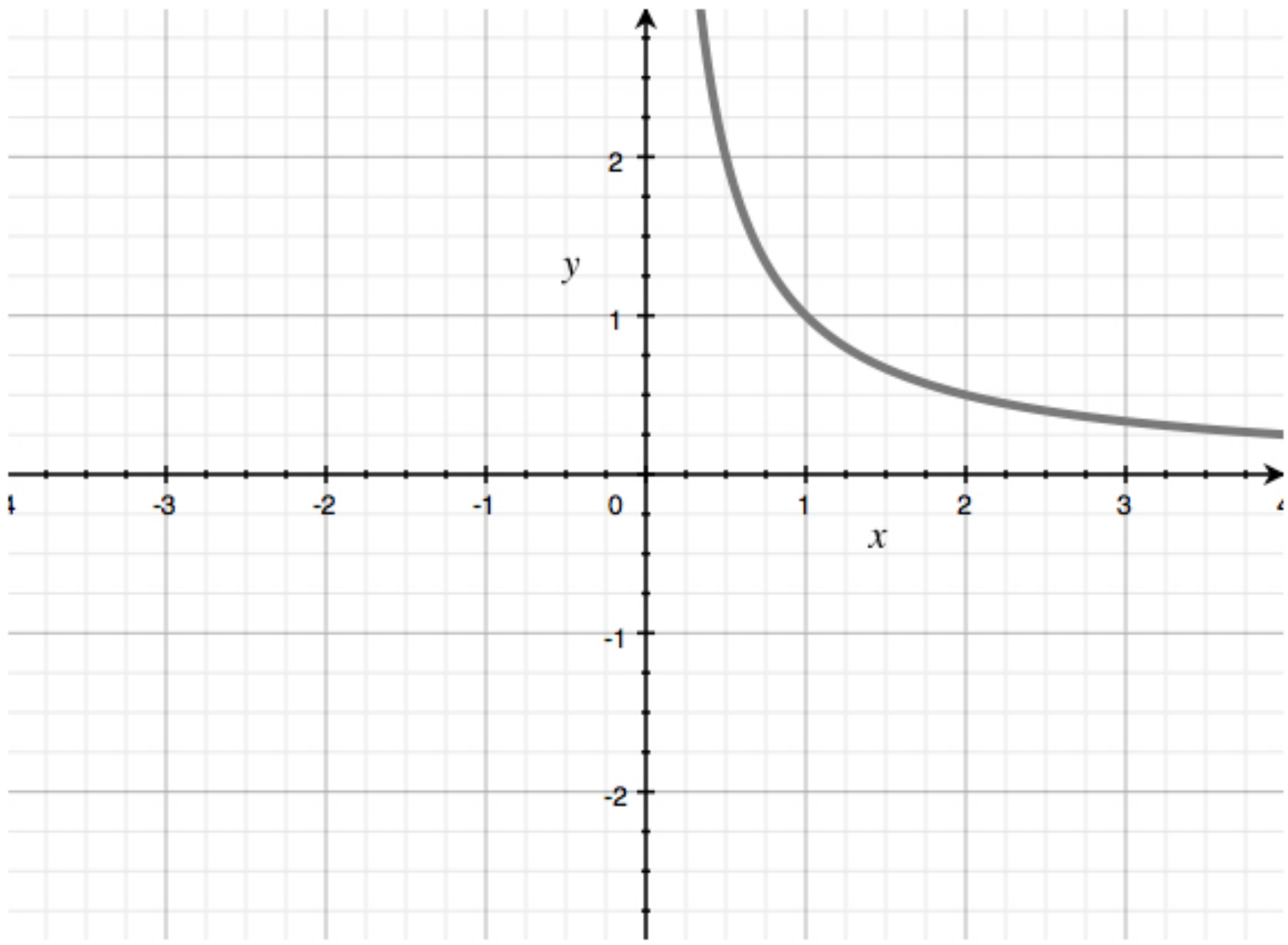
Infinite (essential) discontinuity

You'll see this kind of discontinuity called both infinite discontinuity and essential discontinuity. In either case, it means that the function is discontinuous at a vertical asymptote. Vertical asymptotes are only points of discontinuity when the graph exists on both sides of the asymptote.

The graph below shows a vertical asymptote that makes the graph discontinuous, because the function exists on both sides of the vertical asymptote.



On the other hand, the vertical asymptote in this graph is not a point of discontinuity, because it doesn't break up any part of the graph.



y
x
2
1
-1
-2
-3
-2
-1
0
1
2
3

Derivatives 3

The derivative of a function $f(x)$ is written as $f'(x)$, and read as "f prime of x." By definition, the derivative is the slope of the original function. Let's find out why.

Definition of the derivative

The definition of the derivative, also called the “difference quotient”, is a tool we use to find derivatives “the long way”, before we learn all the shortcuts later that let us find them “the fast way”.

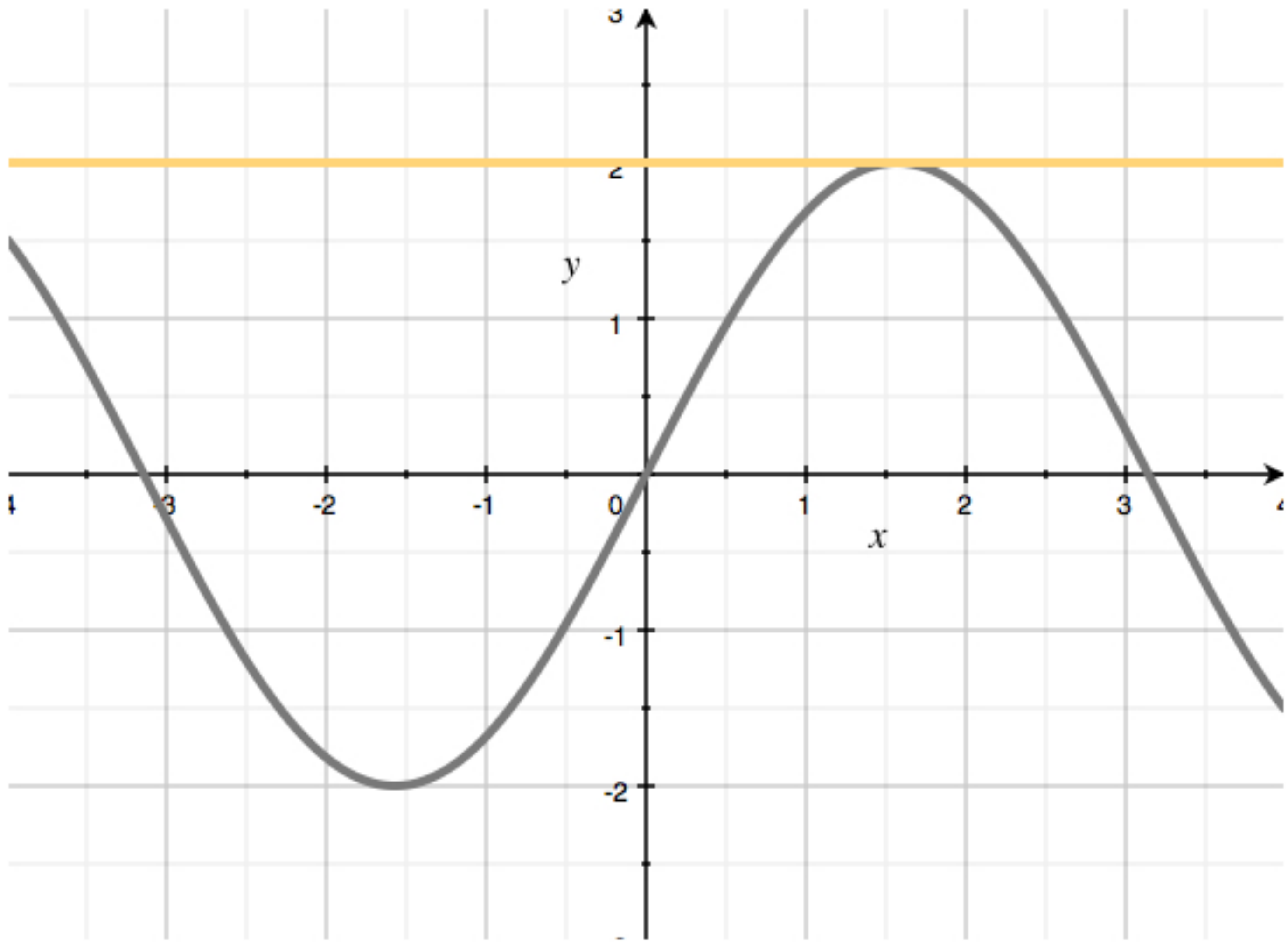
Mostly it’s good to understand the definition of the derivative so that we have a solid foundation for the rest of calculus. So let’s talk about how we build the difference quotient.

Secant and tangent lines

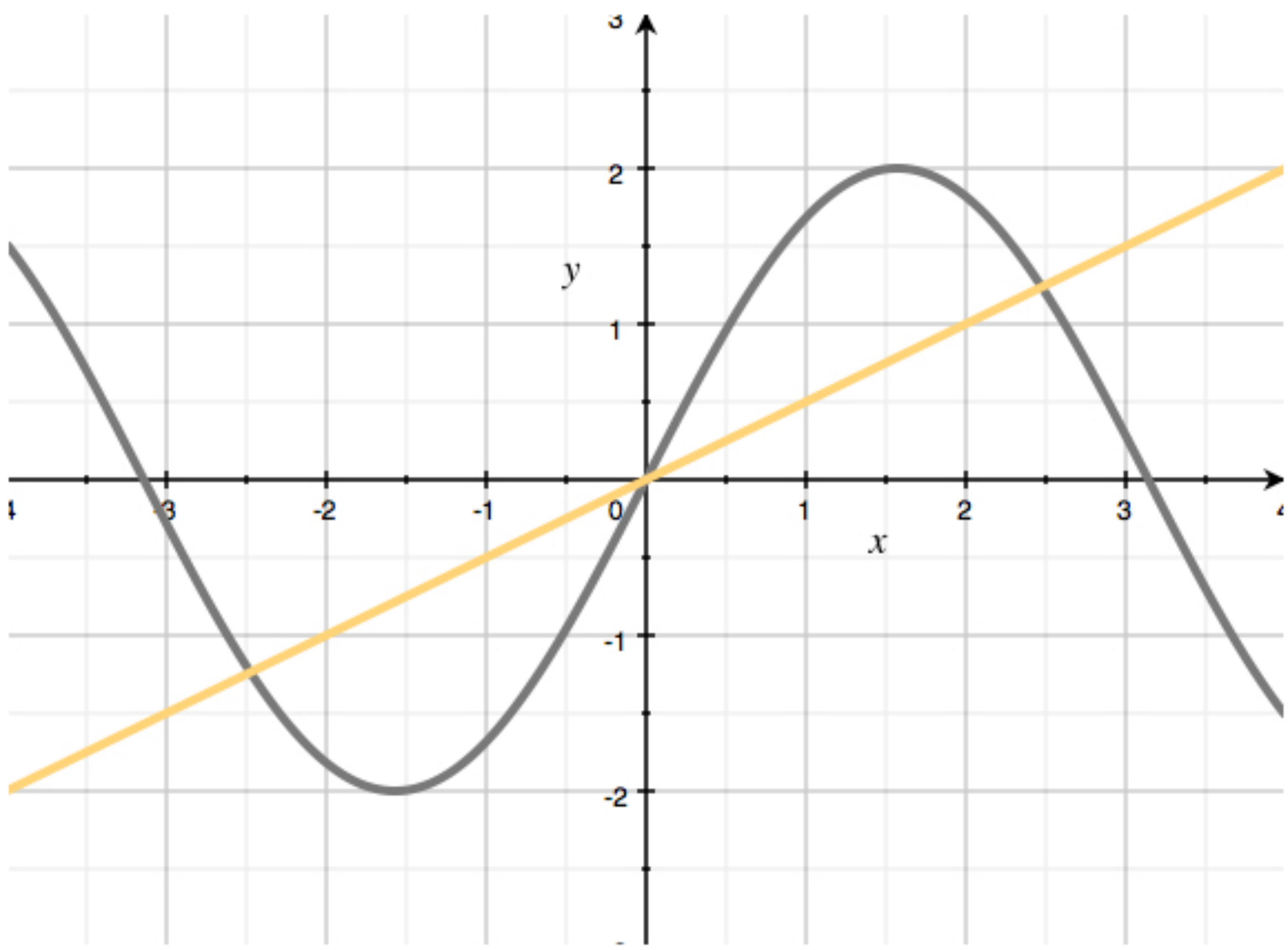
A tangent line is a line that just barely touches the edge of the graph, intersecting it at only one specific point. Tangent lines look very graceful and tidy.

A secant line, on the other hand, is a line that runs right through the middle of a graph, sometimes hitting it at multiple points, and looks generally meaner.

A tangent line:



A secant line:



It’s important to realize here that the slope of the secant line is the average rate of change over the interval between the points where the secant line intersects the graph. The slope of the tangent line instead indicates an instantaneous rate of change, or slope, at the single point where it intersects the graph.

Creating the derivative

If we start with a point $(c, f(c))$ on a graph, and move a certain distance Δx to the right of that point, we can call the new point on the graph $(c + \Delta x, f(c + \Delta x))$.

Connecting those points together gives us a secant line, and we can use

$$\frac{y_2 - y_1}{x_2 - x_1}$$

to determine that the slope of the secant line is

$$\frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}$$

which when we simplify gives us

$$\frac{f(c + \Delta x) - f(c)}{\Delta x}$$

The point is that, if I take my second point and start moving it slowly left, closer to the original point, the slope of the secant line becomes closer to the slope of the tangent line at the original point.

In other words, as the secant line moves closer and closer to the tangent line, the points where the line intersects the graph get closer together, which eventually reduces Δx to 0.

Running through this exercise allows us to realize that if I reduce Δx to 0 and the distance between the two secant points becomes nothing, that the slope of the secant line is now exactly the same as the slope of the tangent line. In fact, we've just changed the secant line into the tangent line entirely.

That's how we create the formula above, which is the very definition of the derivative, which is why the definition of the derivative is the slope of the function at a single point.

Using the difference quotient

To find the derivative of a function using the difference quotient, follow these steps:

1. Plug in $x+h$ for every x in your original function. Sometimes you'll also see h as Δx.

2. Plug your answer from Step 1 in for $f(x+h)$ in the difference quotient.

3. Plug your original function in for $f(x)$ in the difference quotient.

4. Put h in the denominator.

5. Expand all terms and collect like terms.

6. Factor out h from the numerator, then cancel it from the numerator and denominator.

7. Plug in 0 for h and simplify.

Example

Find the derivative.

$$f(x) = x^2 - 5x + 6$$

After replacing x with $(x + \Delta x)$ in $f(x)$, plug in your answer for $f(c + \Delta x)$. Then plug in $f(x)$ as-is for $f(c)$. Put Δx in the denominator.

$$\lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 - 5(x + \Delta x) + 6\right] - (x^2 - 5x + 6)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 5x - 5\Delta x + 6 - x^2 + 5x - 6}{\Delta x}$$

Collect similar terms together then factor Δx out of the numerator and cancel it from the fraction.

$$\lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x - 5\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 2x - 5)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \Delta x + 2x - 5$$

For Δx, plug in the number you're approaching, in this case 0. Then simplify and solve.

$$0 + 2x - 5$$

$$2x - 5$$

Derivative rules

Finally, we've gotten to the point where things start to get easier. We've moved past the difference quotient, which was cumbersome and tedious and generally not fun. You're about to learn several new derivative tricks that will make this whole process a whole lot easier, starting with the derivative of a constant.

The derivative of a constant

The derivative of a constant (a term with no variable attached to it) is always 0. Remember that the graph of any constant is a perfectly horizontal line. Remember also that the slope of any horizontal line is 0. Because the derivative of a function is the slope of that function, and the slope of a horizontal line is 0, the derivative of any constant must be 0.

The derivative rules

In the next sections, we'll learn about how to use the most common derivative rules, including

Power rule	$h(x) = ax^n$	$h'(x) = (a \cdot n)x^{n-1}$
Product rule	$h(x) = f(x)g(x)$	$h'(x) = f(x)g'(x) + f'(x)g(x)$
Quotient rule	$h(x) = \dfrac{f(x)}{g(x)}$	$h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$

Reciprocal rule $h(x) = \frac{a}{f(x)}$ $h'(x) = -a\frac{f'(x)}{[f(x)]^2}$

Power rule

The power rule is the tool you'll use most frequently when finding derivatives. The rule says that for any term of the form ax^n, the derivative of the term is

$$(a \cdot n)x^{n-1}$$

To use the power rule, multiply the variable's exponent n, by its coefficient a, then subtract 1 from the exponent. If there is no coefficient (the coefficient is 1), then the exponent will become the new coefficient.

Example

Find the derivative of the function.

$$f(x) = 7x^3$$

Applying power rule gives

$$f'(x) = 7(3)x^{3-1}$$

$$f'(x) = 21x^2$$

Product rule

If a function contains two variable expressions that are multiplied together, you cannot simply take their derivatives separately and then multiply the derivatives together. You have to use the product rule. Here is the formula:

Given a function

$$h(x) = f(x)g(x)$$

then its derivative is

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

To use the product rule, multiply the first function by the derivative of the second function, then add the derivative of the first function times the second function to your result.

Example

Find the derivative of the function.

$$h(x) = x^2e^{3x}$$

The two functions in this problem are x^2 and e^{3x}. It doesn't matter which one you choose for $f(x)$ and $g(x)$. Let's assign $f(x)$ to x^2 and $g(x)$ to e^{3x}. The derivative of $f(x)$ is $f'(x) = 2x$. The derivative of $g(x)$ is $g'(x) = 3e^{3x}$.

According to the product rule,

$$h'(x) = \left(x^2\right)\left(3e^{3x}\right) + (2x)\left(e^{3x}\right)$$

$$h'(x) = 3x^2e^{3x} + 2xe^{3x}$$

Quotient rule

Just as you must always use the product rule when two variable expressions are multiplied, you must use the quotient rule whenever two variable expressions are divided. Given a function

$$h(x) = \frac{f(x)}{g(x)}$$

then its derivative is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Example

Use quotient rule to find the derivative.

$$h(x) = \frac{x^2}{\ln x}$$

Based on the quotient rule formula, we know that $f(x)$ is the numerator and therefore $f(x) = x^2$ and that $g(x)$ is the denominator and therefore that $g(x) = \ln x$. $f'(x) = 2x$, and $g'(x) = 1/x$. Plugging all of these components into the quotient rule gives

$$h'(x) = \frac{(\ln x)(2x) - (x^2)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$h'(x) = \frac{2x \ln x - x}{\left(\ln x\right)^2}$$

Chain rule with power rule

The chain rule is often one of the hardest concepts for calculus students to understand. It's also one of the most important, and it's used all the time, so make sure you don't leave this section without a solid understanding.

Chain rule lets us calculate derivatives of equations made up of nested functions, where one function is the "outside" function and one function is the "inside function. If we have an equation like

$$y = g\left[f(x)\right]$$

then $g\left[f(x)\right]$ is the outside function and $f(x)$ is the inside function. The derivative looks like

$$y' = \left\{g'\left[f(x)\right]\right\}\left[f'(x)\right]$$

Notice here that we took the derivative first of the outside function, $g\left[f(x)\right]$, leaving the inside function, $f(x)$, completely untouched, and then we multiplied our result by the derivative of the inside function.

So applying the chain rule requires just two simple steps.

1. Take the derivative of the "outside" function, leaving the "inside" function untouched.

2. Multiply your result by the derivative of the "inside" function.

Sometimes it's helpful to use substitution to make it easier to think about $g\left[f(x)\right]$. We just replace the inside function with u, and we get

$$y = g[u]$$

Then the derivative would be

$$y' = g'[u](u')$$

If you're going to use substitution, make sure you back-substitute at the end of the problem to get your final answer.

Example

Use chain rule to find the derivative.

$$y = \left(4x^8 - 6\right)^6$$

Our outside function is $\left(4x^8 - 6\right)^6$, and our inside function is $4x^8 - 6$. Using the substitution method, $u = 4x^8 - 6$ and $u' = 32x^7$.

We'll substitute u into the original equation and get

$$y = (u)^6$$

We'll start to calculate the derivative, and using power rule with chain rule, we find that

$$y' = 6(u)^5(u')$$

Finally, we back-substitute for u and u'.

$$y' = 6\left(4x^8 - 6\right)^5\left(32x^7\right)$$

$$y' = 192x^7 \left(4x^8 - 6\right)^5$$

We just worked an example of chain rule used in conjunction with power rule. We'll also need to know how to use it in combination with product rule, with quotient rule, and with trigonometric functions, which we'll tackle in the next few lessons.

Equation of the tangent line

You'll see it written different ways, but in general the formula for the equation of the tangent line is

$$y = f(a) + f'(a)(x - a)$$

When a problem asks you to find the equation of the tangent line, you'll always be asked to evaluate at the point where the tangent line intersects the graph.

In order to find the equation of the tangent line, you'll need to plug that point into the original function, then substitute your answer for $f(a)$. Next you'll take the derivative of the function, plug the same point into the derivative and substitute your answer for $f'(a)$.

Example

Find the equation of the tangent line at $x = 4$.

$$f(x) = 6x^2 - 2x + 5$$

First, plug $x = 4$ into the original function.

$$f(4) = 6(4)^2 - 2(4) + 5$$

$$f(4) = 96 - 8 + 5$$

$$f(4) = 93$$

Next, take the derivative and plug in $x = 4$.

$$f'(x) = 12x - 2$$

$$f'(4) = 12(4) - 2$$

$$f'(4) = 46$$

Finally, insert both $f(4)$ and $f'(4)$ into the tangent line formula, along with 4 for a, since this is the point at which we're asked to evaluate.

$$y = 93 + 46(x - 4)$$

You can either leave the equation in this form, or simplify it further:

$$y = 93 + 46x - 184$$

$$y = 46x - 91$$

Implicit differentiation

Implicit Differentiation allows you to take the derivative of a function that contains both x and y on the same side of the equation. If you can't solve the function for y, implicit differentiation is the only way to take the derivative.

On the left sides of these derivatives, instead of seeing y' or $f'(x)$, you'll find dy/dx instead. In this notation, the numerator tells you what function you're deriving, and the denominator tells you what variable is being derived. dy/dx is literally read "the derivative of y with respect to x."

One of the most important things to remember, and the thing that usually confuses students the most, is that we have to treat y as a function and not just as a variable like x. Therefore, we always multiply by dy/dx when we take the derivative of y. To use implicit differentiation, follow these steps:

1. Differentiate both sides with respect to x.

2. Whenever you encounter y, treat it as a variable just like x, then multiply that term by dy/dx.

3. Move all terms involving dy/dx to the left side and everything else to the right.

4. Factor out dy/dx on the left and divide both sides by the other left-side factor so that dy/dx is the only thing remaining on the left.

Example

Use implicit differentiation to find the derivative.

$$x^3 + y^3 = 9xy$$

Our first step is to differentiate both sides with respect to x. Treat y as a variable just like x, but whenever you take the derivative of a term that includes y, multiply by dy/dx. You'll need to use the product rule for the right side, treating $9x$ as one function and y as another.

$$3x^2 + 3y^2\frac{dy}{dx} = (9)(y) + (9x)(1)\frac{dy}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

Move all terms that include dy/dx to the left side, and move everything else to the right side.

$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

Factor out dy/dx on the left, then divide both sides by $\left(3y^2 - 9x\right)$.

$$\frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

Dividing the right side by 3 to simplify gives us our final answer.

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

Equation of the tangent line

We'll do another complete example in the next section, but let's get a preview of what it looks like to find the equation of the tangent line for an implicitly-defined function.

Just for fun, let's pretend you're asked to find the equation of the tangent line of the function in the previous example at the point $(2,3)$.

You'd just pick up right where you left off, and plug in this point to the derivative of the function to find the slope of the tangent line.

Example (continued)

$$\frac{dy}{dx}(2,3) = \frac{3(3) - (2)^2}{(3)^2 - 3(2)}$$

$$\frac{dy}{dx}(2,3) = \frac{9 - 4}{9 - 6}$$

$$\frac{dy}{dx}(2,3) = \frac{5}{3}$$

Since you have the point $(2,3)$ and the slope of the tangent line at the point $(2,3)$, plug the point and the slope into point-slope form to find the equation of the tangent line. Then simplify.

$$y - 3 = \frac{5}{3}(x - 2)$$

$$3y - 9 = 5(x - 2)$$

$$3y - 9 = 5x - 10$$

$$3y = 5x - 1$$

$$y = \frac{5}{3}x - \frac{1}{3}$$

Optimization

Graph sketching is not very hard, but there are a lot of steps to remember. Like anything, the best way to master it is with a lot of practice.

When it comes to sketching the graph, if possible I absolutely recommend graphing the function on your calculator before you get started so that you have a visual of what your graph should look like when it's done. You certainly won't get all the information you need from your calculator, so unfortunately you still have to learn the steps, but it's a good double-check system.

Our strategy for sketching the graph will include the following steps:

1. Find critical points.

2. Determine where $f(x)$ is increasing and decreasing.

3. Find inflection points.

4. Determine where $f(x)$ is concave up and concave down.

5. Find x- and y-intercepts.

6. Plot critical points, possible inflection points and intercepts.

7. Determine behavior as $f(x)$ approaches $\pm\infty$.

8. Draw the graph with the information we've gathered.

Critical points

Critical points occur at x-values where the function's derivative is either equal to zero or undefined. Critical points are the only points at which a function can change direction, and also the only points on the graph that can be maxima or minima of the function.

Example

Find the critical points of the function.

$$f(x) = x + \frac{4}{x}$$

Take the derivative and simplify. You can move the x in the denominator of the fraction into the numerator by changing the sign on its exponent from 1 to -1.

$$f(x) = x + 4x^{-1}$$

Using power rule to take the derivatives gives

$$f'(x) = 1 - 4x^{-2}$$

$$f'(x) = 1 - \frac{4}{x^2}$$

Now set the derivative equal to 0 and solve for x.

$$0 = 1 - \frac{4}{x^2}$$

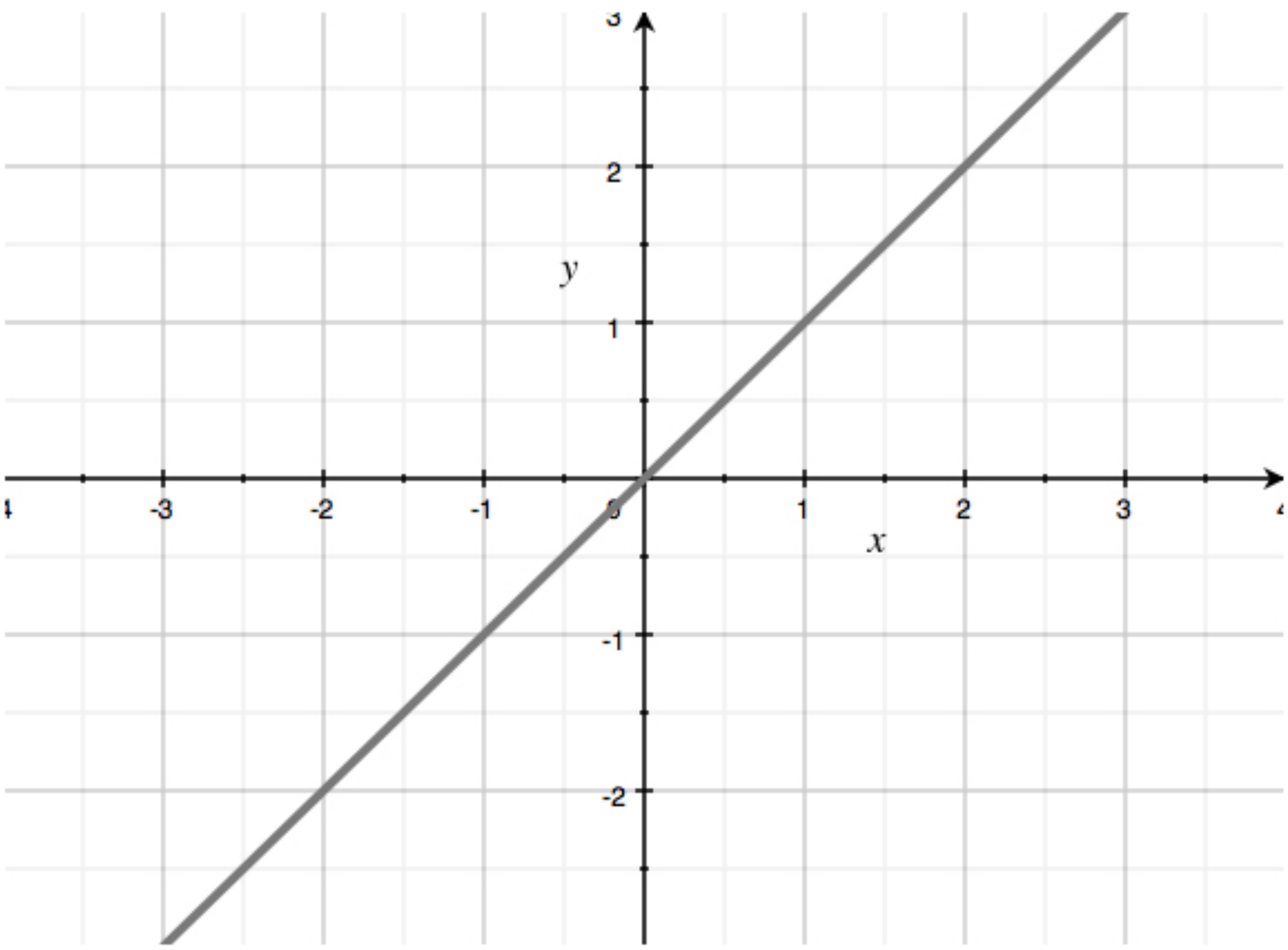
$$1 = \frac{4}{x^2}$$

$$x^2 = 4$$

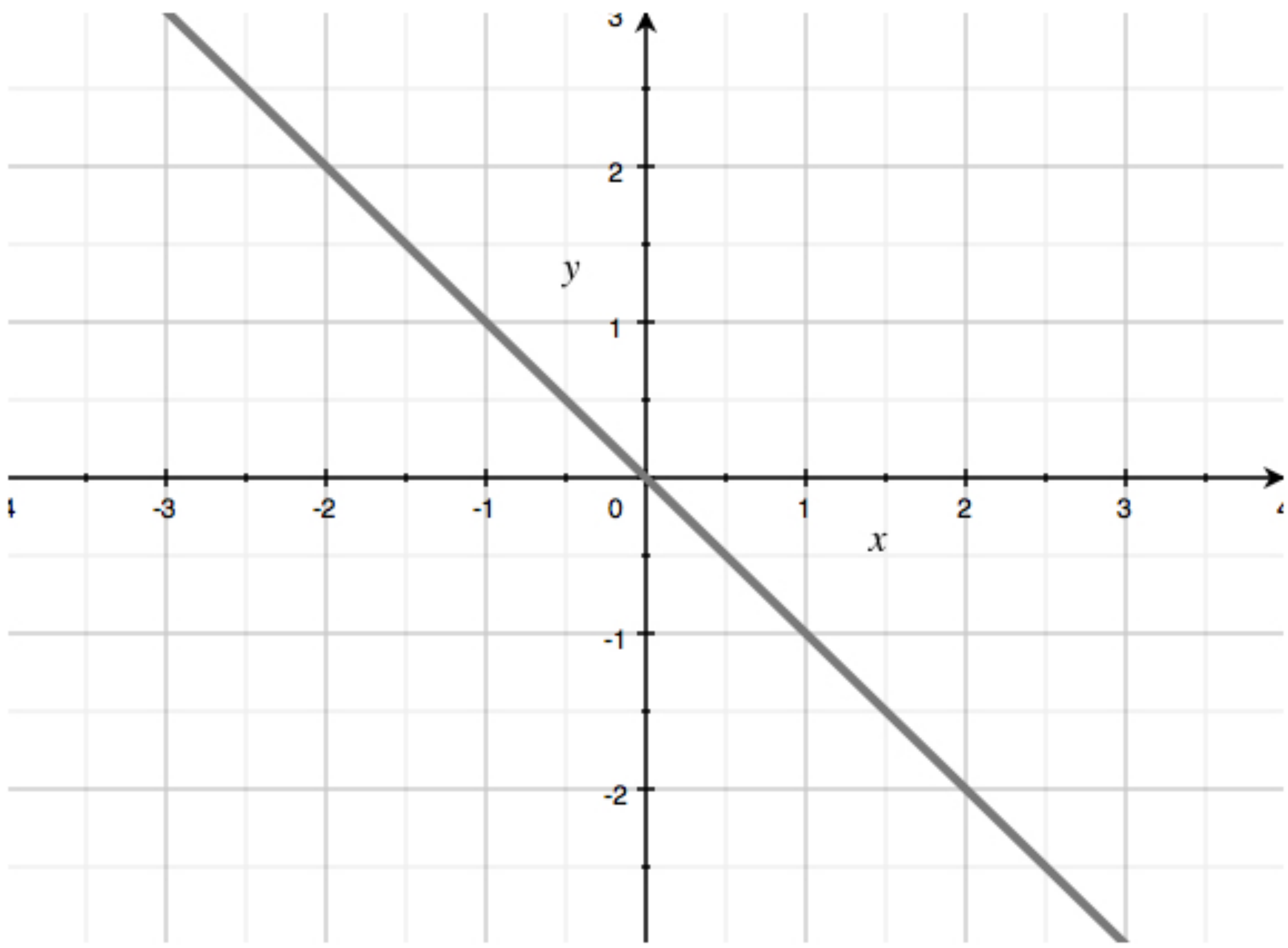
$$x = \pm 2$$

Increasing and decreasing

A function that is increasing (moving up as you travel from left to right along the graph), has a positive slope, and therefore a positive derivative.



Similarly, a function that is decreasing (moving down as you travel from left to right along the graph), has a negative slope, and therefore a negative derivative.



Based on this information, it makes sense that the sign (positive or negative) of a function's derivative indicates the direction of the original function. If the derivative is positive at a point, the original function is increasing at that point. Not surprisingly, if the derivative is negative at a point, the original function is decreasing there.

We already know that the direction of the graph can only change at the critical points that we found earlier. As we continue with our example, we'll therefore plot those critical points on a wiggle graph to test where the function is increasing and decreasing.

Example (continued)

Find the intervals on which the function is increasing and decreasing.

$$f(x) = x + \frac{4}{x}$$

First, we create our wiggle graph and plot our critical points.

Next, we pick values on each interval of the wiggle graph and plug them into the derivative. If we get a positive result, the graph is increasing. A negative result means it’s decreasing. The intervals that we will test are

$$-\infty < x < -2$$

$$-2 < x < 2$$

$$2 < x < \infty$$

To test $-\infty < x < -2$, we’ll plug -3 into the derivative, since -3 is a value in that interval.

$$f'(-3) = 1 - \frac{4}{(-3)^2}$$

$$f'(-3) = 1 - \frac{4}{9}$$

$$f'(-3) = \frac{9}{9} - \frac{4}{9}$$

$$f'(-3) = \frac{5}{9} > 0$$

To test $-2 < x < 2$, we'll plug -1 into the derivative.

$$f'(-1) = 1 - \frac{4}{(-1)^2}$$

$$f'(-1) = 1 - 4$$

$$f'(-1) = -3 < 0$$

To test $2 < x < \infty$, we'll plug 3 into the derivative.

$$f'(3) = 1 - \frac{4}{(3)^2}$$

$$f'(3) = 1 - \frac{4}{9}$$

$$f'(3) = \frac{9}{9} - \frac{4}{9}$$

$$f'(3) = \frac{5}{9} > 0$$

Now we plot the results on our wiggle graph,

and we can see that $f(x)$ is

- increasing on $-\infty < x < -2$
- decreasing on $-2 < x < 2$
- increasing on $2 < x < \infty$

Inflection points

Inflection points are just like critical points, except that they indicate where the graph changes concavity, instead of indicating where the graph changes direction, which is the job of critical points. We'll learn about concavity in the next section. For now, let's find our inflection points.

In order to find inflection points, we first take the second derivative, which is the derivative of the derivative. We then set the second derivative equal to 0 and solve for x.

Example (continued)

We'll start with the first derivative, and take its derivative to find the second derivative.

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(x) = 1 - 4x^{-2}$$

$$f''(x) = 0 + 8x^{-3}$$

$$f''(x) = \frac{8}{x^3}$$

Now set the second derivative equal to 0 and solve for x.

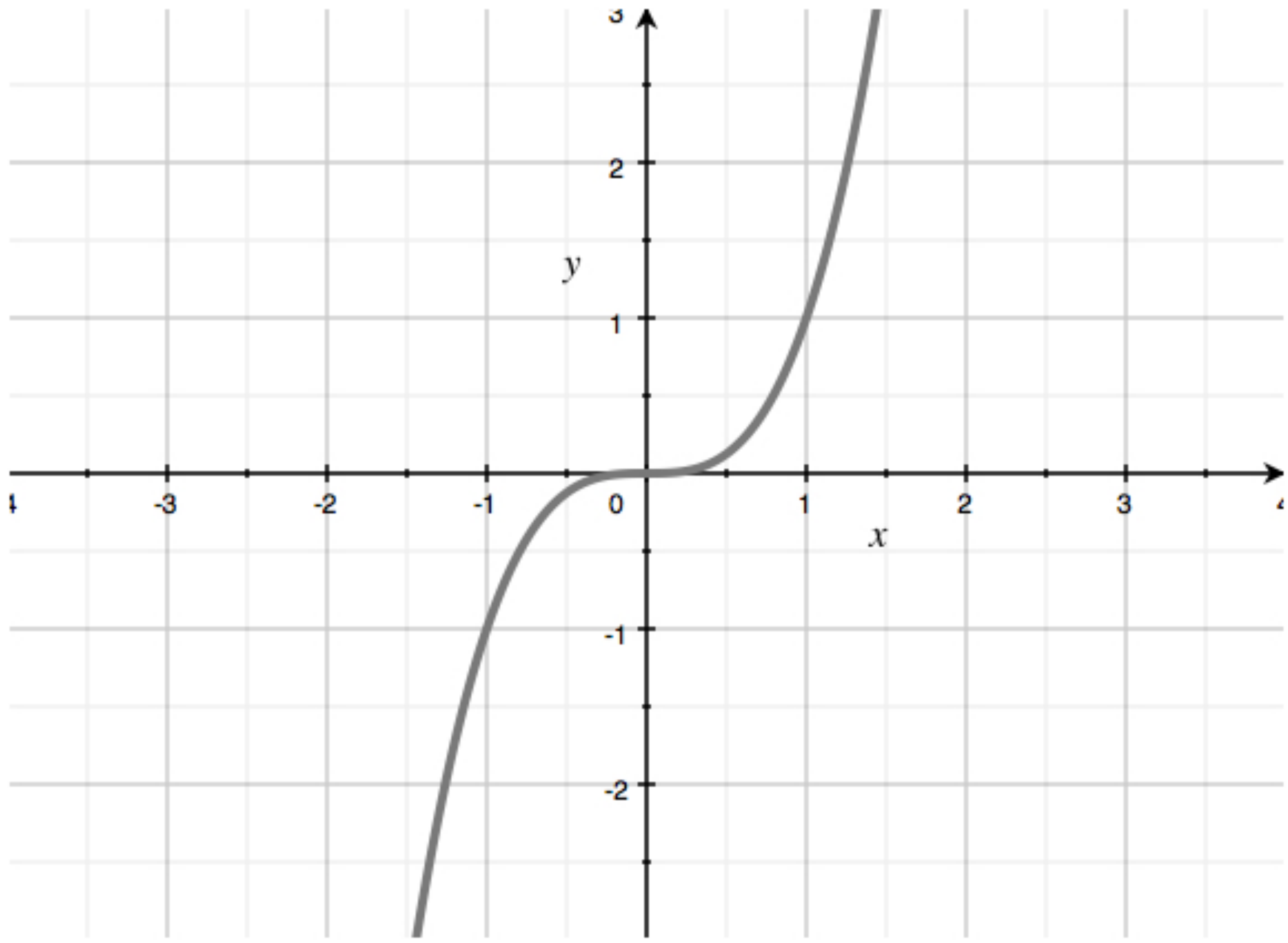
$$0 = \frac{8}{x^3}$$

There is no solution to this equation, but we can see that the second derivative is undefined at $x = 0$. Therefore, $x = 0$ is the only possible inflection point.

Concavity

Concavity is indicated by the sign of the function's second derivative, $f''(x)$. The function is concave up everywhere the second derivative is positive ($f''(x) > 0$), and concave down everywhere the second derivative is negative ($f''(x) < 0$).

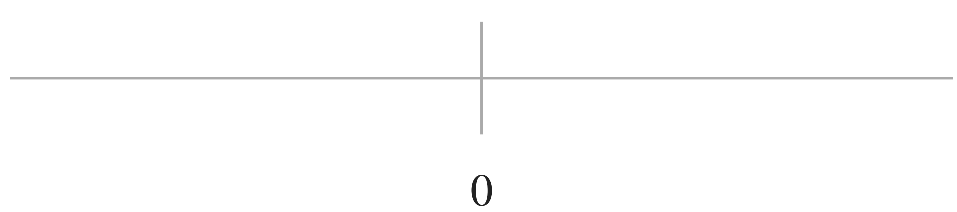
The graph below illustrates examples of concavity. From $-\infty < x < 0$, the graph is concave down. Think about the fact that a graph that is concave down looks like a frown. The inflection point at which the graph changes concavity is at $x = 0$. On the interval $0 < x < \infty$, the graph is concave up, and it looks like a smile.



We can use the same wiggle graph technique, along with the possible inflection point we just found, to test for concavity.

Example (continued)

Since our only inflection point was at $x = 0$, let's go ahead and plot that on our wiggle graph now.



As you might have guessed, we'll be testing values in the following intervals:

$$-\infty < x < 0$$

$$0 < x < \infty$$

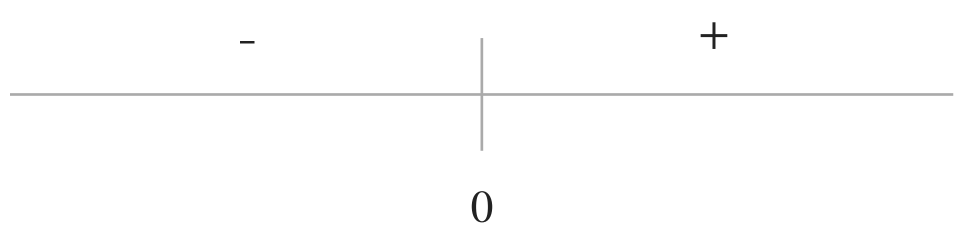
To test $-\infty < x < 0$, we'll plug -1 into the second derivative.

$$f''(-1) = \frac{8}{(-1)^3} = -8 < 0$$

To test $0 < x < \infty$, we'll plug 1 into the second derivative.

$$f''(1) = \frac{8}{(1)^3} = 8 > 0$$

Now we can plot the results on our wiggle graph.



We determine that $f(x)$ is concave down on the interval $-\infty < x < 0$ and concave up on the interval $0 < x < \infty$.

Intercepts

To find the points where the graph intersects the x- and y-axis, we can plug 0 into the original function for one variable and solve for the other.

Example (continued)

Given our original function

$$f(x) = x + \frac{4}{x}$$

we'll plug 0 in for x to find y-intercepts.

$$y = 0 + \frac{4}{0}$$

Immediately we can recognize there are no y-intercepts because we can't have a 0 result in the denominator.

Let's plug in 0 for y to find x-intercepts.

$$0 = x + \frac{4}{x}$$

Multiply every term by x to eliminate the fraction.

$$0 = x^2 + 4$$

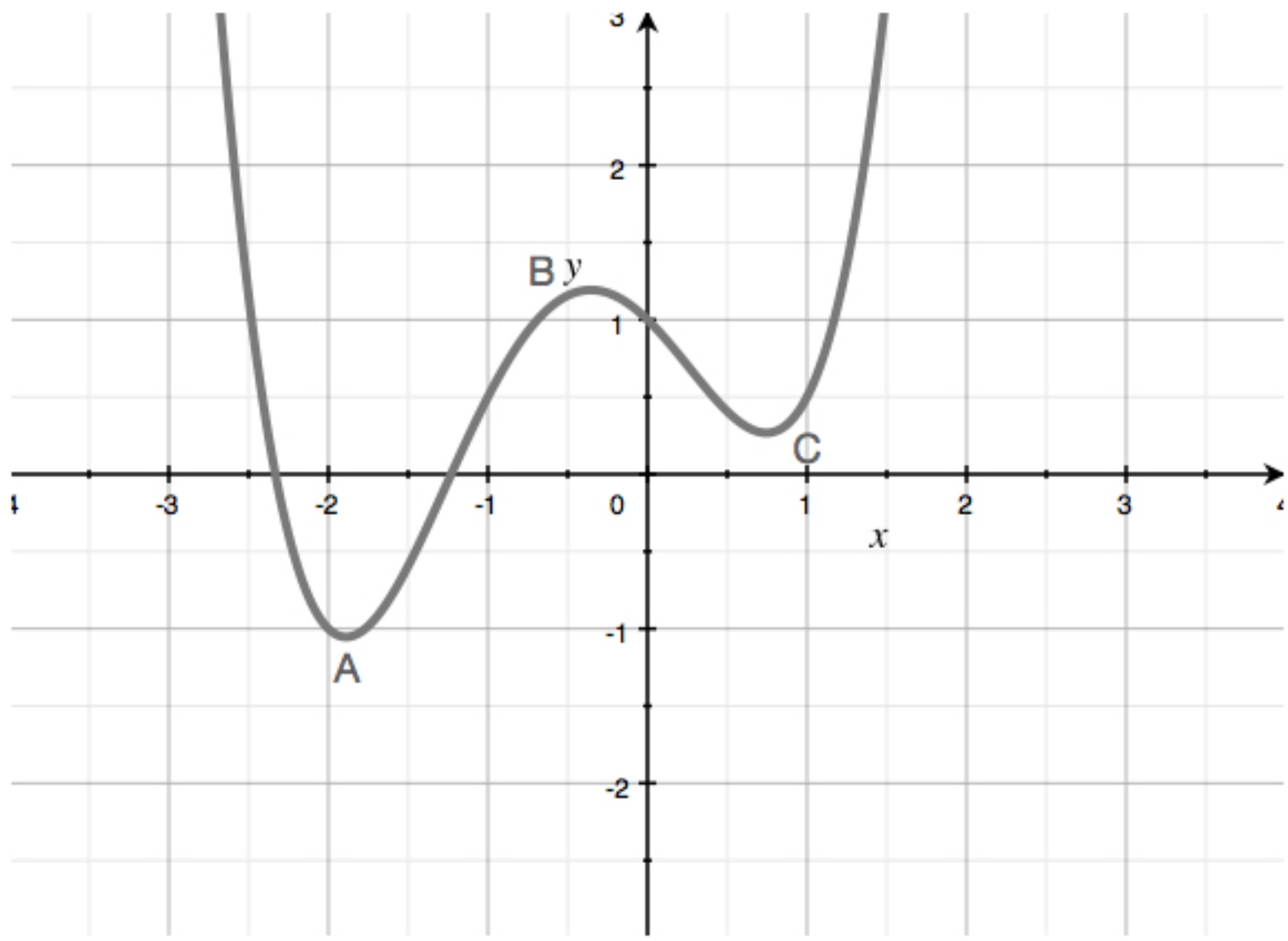
$$-4 = x^2$$

Since there are no real solutions to this equation, we know that this function has no x-intercepts.

Local and global extrema

Maxima and minima (these are the plural versions of the singular words maximum and minimum) can only exist at critical points, but not every critical point is necessarily an extrema. To know for sure, you have to test each solution separately.

In the graph below, minimums exist at points A and C. Based on the y-values at those points, the global minimum exists at A, and a local minimum exists at C.



If you're dealing with a closed interval, for example some function $f(x)$ on the interval $[0,5]$, then the endpoints at $x = 0$ and $x = 5$ are candidates for extrema and must also be tested. We'll use the first derivative test to find extrema.

First derivative test

Remember the wiggle graph that we created from our earlier test for increasing and decreasing?

Based on the positive and negative signs on the graph, you can see that the function is increasing, then decreasing, then increasing again, and if you can picture a function like that in your head, then you know immediately that we have a local maximum at $x = -2$ and a local minimum at $x = 2$.

You really don't even need the first derivative test, because it tells you in a formal way exactly what you just figured out on your own:

1. If the derivative is negative to the left of the critical point and positive to the right of it, the graph has a local minimum at that point.

2. If the derivative is positive to the left of the critical point and negative on the right side of it, the graph has a local maximum at that point.

As a side note, if it's positive on both sides or negative on both sides, then the point is neither a local maximum nor a local minimum, and the test is inconclusive.

Remember, if you have more than one local maximum or minimum, you must plug in the value of x at the critical points to your original function. The y-values you get back will tell you which points are global maxima and minima, and which ones are only local. For example, if you find that your function has two local maxima, you can plug in the value for x at those critical points. As an example, if the first returns a y-value of 10 and the second returns a y-value of 5, then the first point is your global maximum and the second point is your local maximum.

If you're asked to determine where the function has its maximum/minimum, your answer will be in the form x=[value]. But if you're asked for the value at the maximum/minimum, you'll have to plug the x-value into your original function and state the y-value at that point as your answer.

Second derivative test

You can also test for local maxima and minima using the second derivative test if it easier for you than the first derivative test. In order to use this test, simply plug in your critical points to the second derivative. If your result is negative, that point is a local maximum. If the result is positive, the point is a local minimum. If the result is zero, you can't draw a conclusion from the second derivative test, and you have to resort to the first derivative test to solve the problem. Let's try it.

Example (continued)

Remember that our critical points are $x=-2$ and $x=2$.

$$f''(-2)=\frac{8}{(-2)^3}$$

$$f''(-2)=\frac{8}{-8}$$

$$f''(-2)=-1<0$$

Since the second derivative is negative at $x = -2$, we conclude that there is a local maximum at that point.

$$f''(2) = \frac{8}{(2)^3}$$

$$f''(2) = \frac{8}{8}$$

$$f''(2) = 1 > 0$$

Since the second derivative is positive at $x = 2$, we conclude that there is a local minimum at that point.

These are the same results we got from the first derivative test, so why did we do this? Because you may be asked on a test to use a particular method to test the extrema, so you should really know how to use both tests.

Vertical asymptotes

Vertical asymptotes are the easiest to test for, because they only exist where the function is undefined. Remember, a function is undefined whenever we have a value of zero as the denominator of a fraction, or whenever we have a negative value inside a square root sign. Consider the example we've been working with in this section:

$$f(x) = x + \frac{4}{x}$$

You should see immediately that we have a vertical asymptote at $x = 0$ because plugging in 0 for x makes the denominator of the fraction 0, and therefore undefined.

Horizontal asymptotes

Vertical and horizontal asymptotes are similar in that they can only exist when the function is a rational function.

When we're looking for horizontal asymptotes, we only care about the first term in the numerator and denominator. Both of those terms will have what's called a degree, which is the exponent on the variable. If our function is

$$f(x) = \frac{x^3 + \text{lower-degree terms}}{x^2 + \text{lower-degree terms}}$$

then the degree of the numerator is 3 and the degree of the denominator is 2.

Here's how we test for horizontal asymptotes.

1. If the degree of the numerator is less than the degree of the denominator, then the x-axis is a horizontal asymptote.

2. If the degree of the numerator is equal to the degree of the denominator, then the coefficient of the first term in the numerator divided by the coefficient in the first term of the denominator is the horizontal asymptote.

3. If the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

Using the example we've been working with throughout this section, we'll determine whether the function has any horizontal asymptotes. We can use long division to convert the function into one fraction. The following is the same function as our original function, just consolidated into one fraction after multiplying the first term by x/x:

$$f(x) = \frac{x^2 + 4}{x}$$

We can see immediately that the degree of our numerator is 2, and that the degree of our denominator is 1. That means that our numerator is one degree higher than our denominator, which means that this function does not have a horizontal asymptote.

Slant asymptotes

Slant asymptotes are a special case. They exist when the degree of the numerator is 1 greater than the degree of the denominator. Let's take the example we've been using throughout this section.

$$f(x) = x + \frac{4}{x}$$

First, we'll convert this function to a rational function by multiplying the first term by x/x and then combining the fractions.

$$f(x) = \frac{x^2 + 4}{x}$$

We can see that the degree of our numerator is one greater than the degree of our denominator, so we know that we have a slant asymptote.

To find the equation of that asymptote, we divide the denominator into the numerator using long division and we get

$$f(x) = x + \frac{4}{x}$$

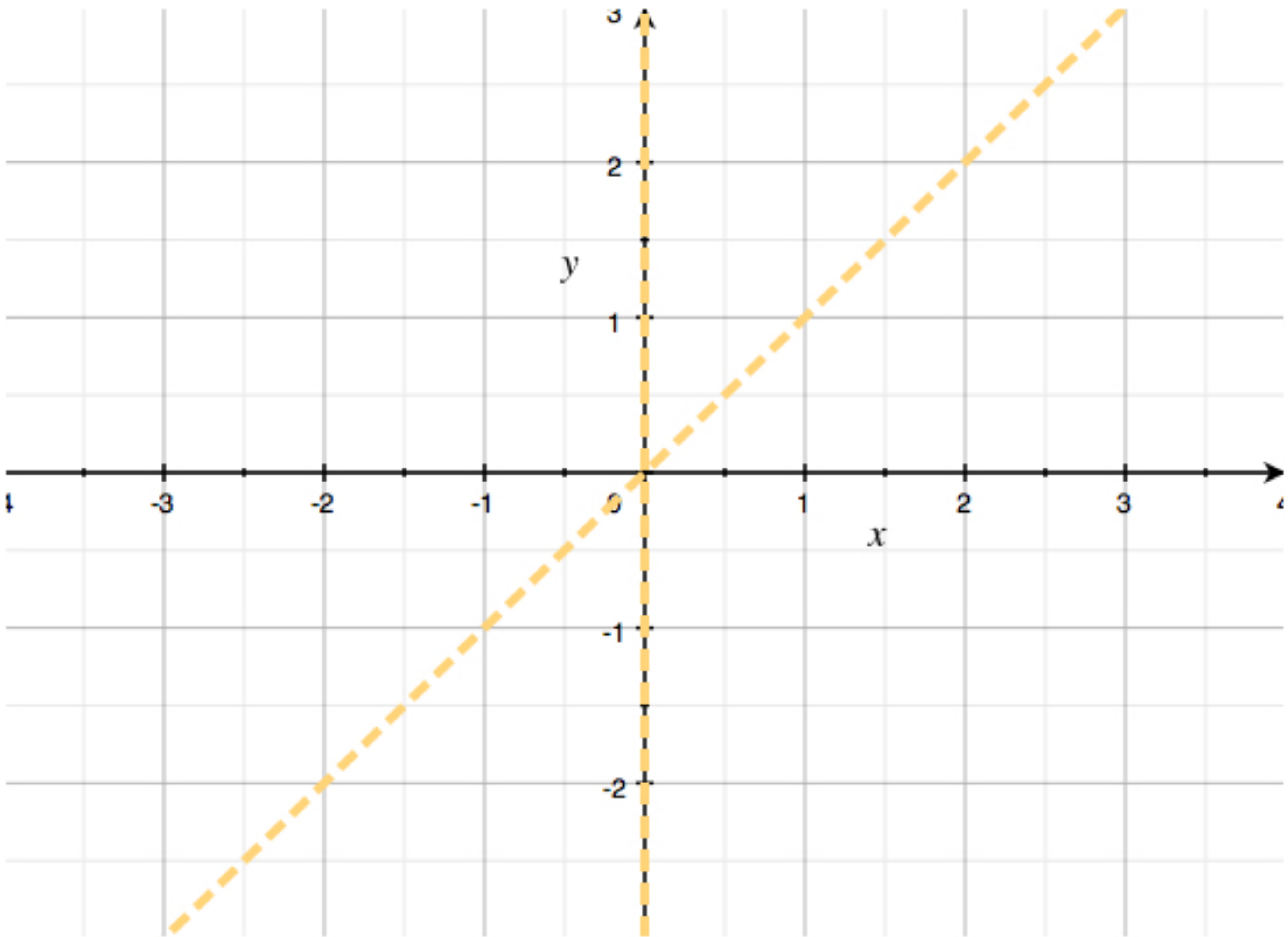
Right back to our original function! That won't always happen, our function just happened to be the composition of the quotient and remainder.

Whenever we use long division in this way to find the slant asymptote, the polynomial part is the slant asymptote and the denominator of the fraction gives the vertical asymptote. Therefore, in this case, our slant asymptote is the line $y = x$, and the vertical asymptote is the line $x = 0$.

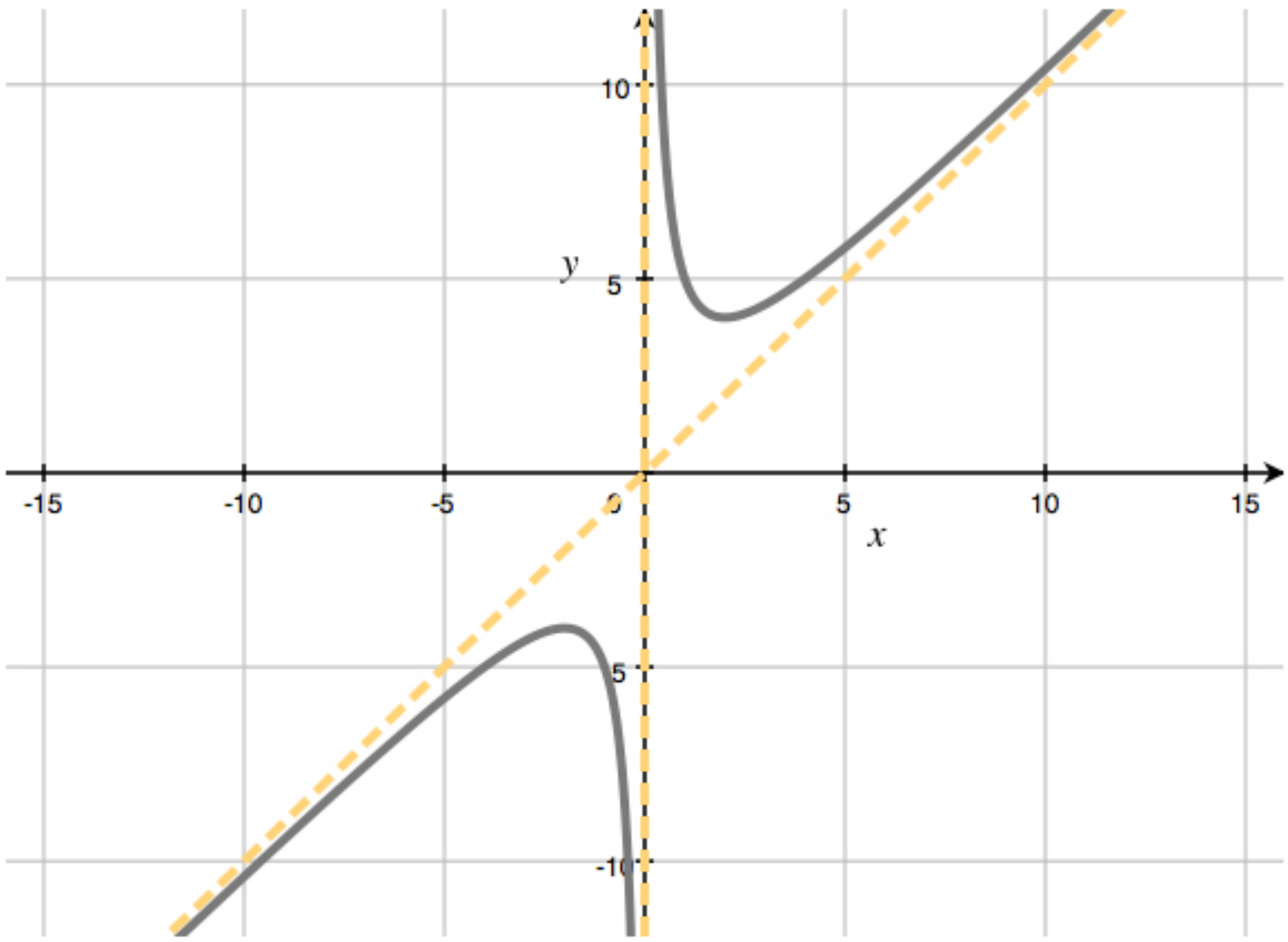
Sketching the graph

Now that we've finished gathering all of the information we can about our graph, we can start sketching it. This will be something you'll just have to practice and get the hang of.

The first thing you should do is sketch any asymptotes, because you know that your graph won't cross those lines, so they act as good guidelines. So let's draw in the lines $x = 0$ and $y = x$.



Knowing that the graph is concave up in the upper right, and concave down in the lower left, and realizing that it can’t cross either of the asymptotes, you should be able to make a pretty good guess that it looks like:



In this case, picturing the graph was a little easier because of the two asymptotes, but if you didn't have the slant asymptote, you'd want to graph the x- and y-intercepts, critical and inflection points, and extrema, and then connect the points using the information you have about increasing/decreasing and concavity.

Integrals

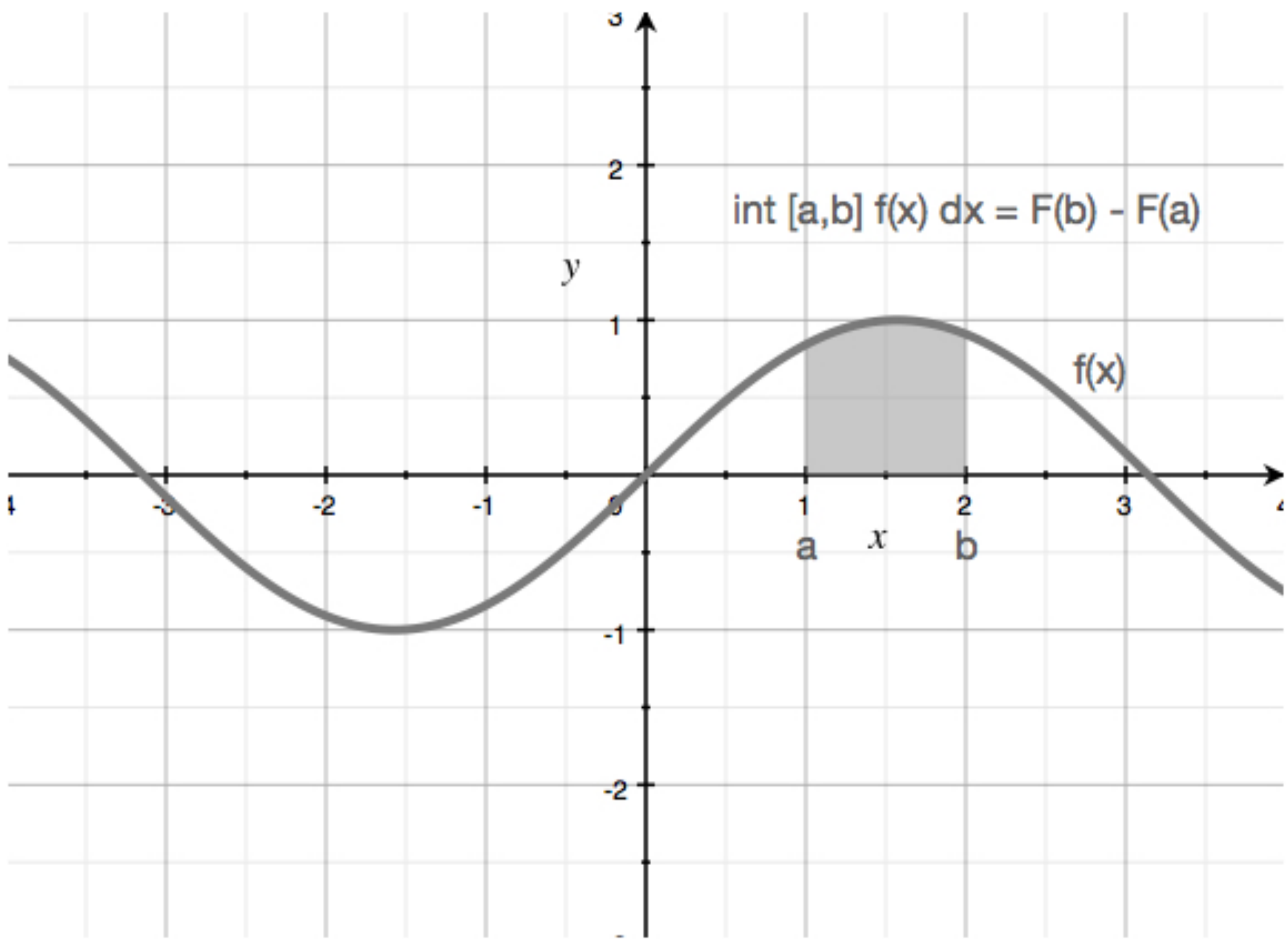
4

The integral of a function is its antiderivative. In other words, to find a function's integral, we perform the opposite actions that we would have taken to find its derivative. The value we find for the integral models the area underneath the graph of the function. Let's find out why.

Definite integrals

Evaluating a definite integral means finding the area enclosed by the graph of the function and the x-axis, over the given interval $[a, b]$.

In the graph below, the shaded area is the integral of $f(x)$ on the interval $[a, b]$. Finding this area means taking the integral of $f(x)$, plugging the upper limit b into the result, and then subtracting from that whatever you get when you plug in the lower limit a.



Example

Evaluate the integral.

$$\int_0^2 3x^2 - 5x + 2\ dx$$

If we let $f(x) = 3x^2 - 5x + 2$ and then integrate the polynomial, we get

$$F(x) = \left(x^3 - \frac{5}{2}x^2 + 2x + C \right) \Big|_0^2$$

where C is the constant of integration.

Evaluating on the interval $[0,2]$, we get

$$F(x) = \left[(2)^3 - \frac{5}{2}(2)^2 + 2(2) + C \right] - \left[(0)^3 - \frac{5}{2}(0)^2 + 2(0) + C \right]$$

$$F(x) = (8 - 10 + 4 + C) - (0 - 0 + 0 + C)$$

$$F(x) = 8 - 10 + 4 + C - C$$

$$F(x) = 2$$

As you can see, the constant of integration "cancels out" in the end, leaving a definite value as the final answer, not just a function for y defined in terms of x.

Since this will always be the case, you can just leave C out of your answer whenever you're solving a definite integral.

So, what do we mean when we say $F(x) = 2$? What does this value represent? When we say that $F(x) = 2$, it means that the area

1. below the graph of $f(x)$,

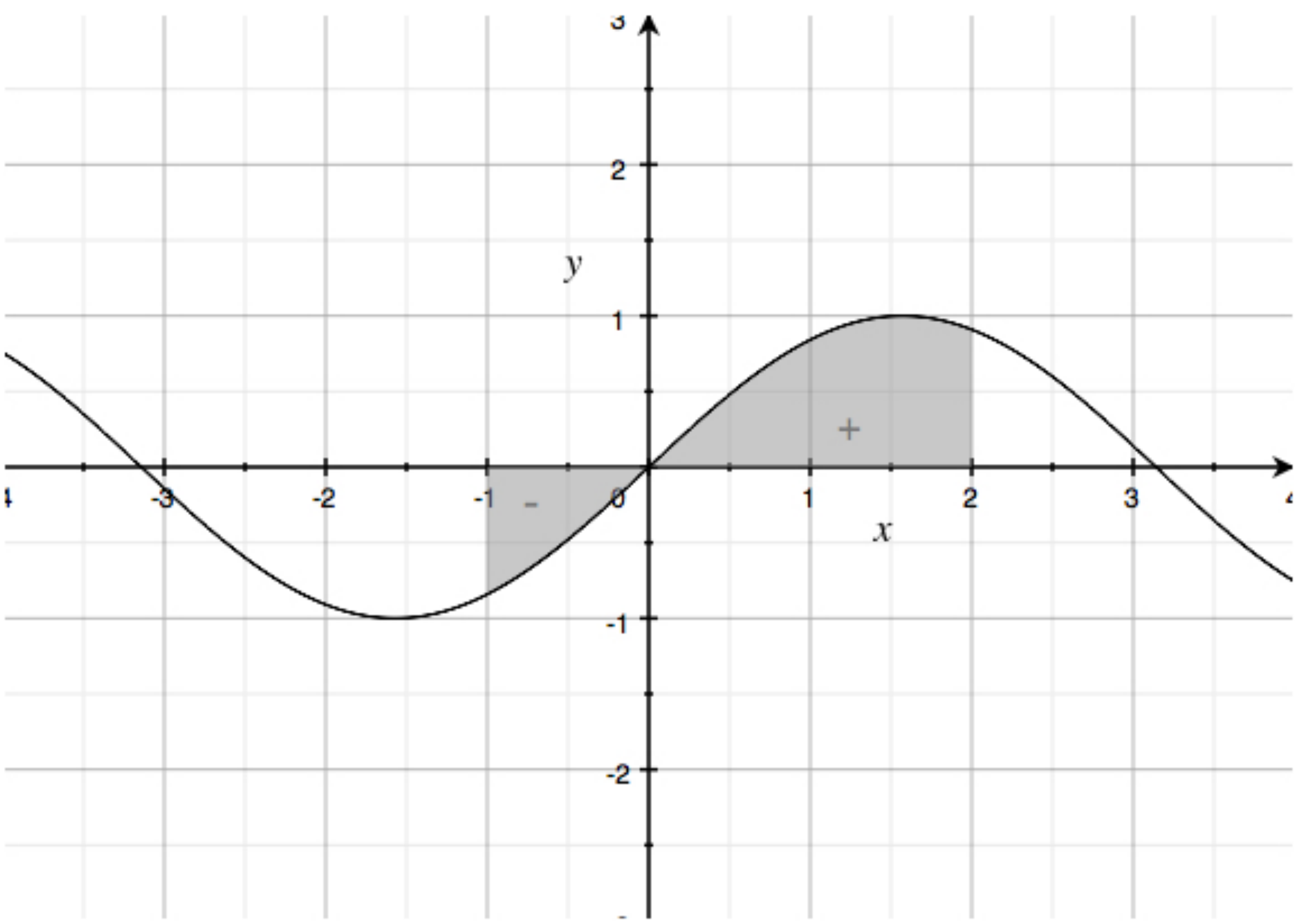
2. above the x-axis, and

3. between the lines $x = 0$ and $x = 2$

is 2 square units.

Keep in mind that we're talking about the area enclosed by the graph and the x-axis. If $f(x)$ drops below the x-axis inside $[a, b]$, we treat the area under the x-axis as negative area.

Then finding the value of $F(x)$ means subtracting the area enclosed by the graph under the x-axis from the area enclosed by the graph above the x-axis.



In other words, evaluating the definite integral of $f(x) = \sin x$ on $[-1,2]$ means subtracting the area enclosed by the graph below the x-axis from the area enclosed by the graph above the x-axis.

This means that, if the area enclosed by the graph below the x-axis is larger than the area enclosed by the graph above the x-axis, then the value of $F(x)$ will be negative ($F(x) < 0$).

If the area enclosed by the graph below the x-axis is exactly equal to the area enclosed by the graph above the x-axis, then $F(x) = 0$.

Fundamental theorem of calculus part 1

The fundamental theorem of calculus (FTC) is the formula that relates the derivative to the integral and provides us with a method for evaluating definite integrals.

Part 1

Part 1 of the Fundamental Theorem of Calculus states that

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$, which means that the integral of $f(x)$ on the interval $[a, b]$ is $F(x)$.

Part 1 of the FTC tells us that we can figure out the exact value of an indefinite integral (area under the curve) when we know the interval over which to evaluate (in this case the interval $[a, b]$).

There are rules to keep in mind. First, the function $f(x)$ must be continuous during the the interval in question. This means that between a and b the graph of the function cannot have any breaks (where it does not exist), holes (where it does not exist at a single point) or jumps (where the function exists at two separate y-values for a single x-value). Second, the interval must be closed, which means that both limits must be constants (real numbers only, no infinity allowed).

When it comes to solving a problem using Part 1 of the Fundamental Theorem, we can use the chart below to help us figure out how to do it.

Given integral	**How to solve it**
$f(x)=\int_a^x f(t)\ dt$	Plug x in for t.
$f(x)=\int_x^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug x in for t.
$f(x)=\int_a^{g(x)} f(t)\ dt$	Plug $g(x)$ in for t, then multiply by dg/dx.
$f(x)=\int_{g(x)}^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.
$f(x)=\int_{g(x)}^{h(x)} f(t)\ dt$	Split the limits of integration as $\int_{g(x)}^0 f(t)\ dt+\int_0^{h(x)} f(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^0 f(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Example

Use Part 1 of the Fundamental Theorem of Calculus to find the value of the integral.

$$F(x)=\int_1^3 x^3\ dx$$

First, we perform the integration.

$$F(x)=\frac{x^4}{4}\bigg|_1^3$$

Next, we plug in the upper and lower limits, subtracting the lower limit from the upper limit.

$$F=\frac{(3)^4}{4}-\frac{(1)^4}{4}$$

$$F=\frac{81}{4}-\frac{1}{4}$$

$$F=\frac{80}{4}$$

Let's double check that this satisfies Part 1 of the FTC.

If we break the equation into parts,

$$F(b)=\int x^3\ dx \text{ where } b=3 \text{ and } F(a)=\int x^3\ dx \text{ where } a=1$$

and evaluate the two equations separately, we can double check our answer.

First we integrate as an indefinite integral.

$$F(x) = \int x^3\ dx$$

$$F(x) = \frac{x^4}{4} + C$$

Next we plug in $b = 3$ and $a = 1$.

$$F(3) = \frac{(3)^4}{4} + C$$

$$F(1) = \frac{(1)^4}{4} + C$$

Finally, we find $F(b) - F(a)$.

$$F(3) - F(1) = \frac{(3)^4}{4} + C - \left[\frac{(1)^4}{4} + C\right]$$

$$F(3) - F(1) = \frac{(3)^4}{4} + C - \frac{(1)^4}{4} - C$$

$$F(3) - F(1) = \frac{80}{4}$$

As you can see, we've verified that value of F that we found earlier. This answer is what we expected and it confirms Part 1 of the FTC.

Fundamental theorem of calculus part 2

Part 2 of the Fundamental Theorem of Calculus states that

If $F(x) = \int_a^x f(t)\ dt$,

where $f(t)$ is continuous,

then $\dfrac{dF(x)}{dx} = f(x)$

This means that if you take the integral of the function $f(t)$ over the interval $[a, x]$, the answer you get can be derived to get back to $f(x)$. What this means is that you can double check your integration for mistakes. In order for this to work, the interval you're evaluating must include one variable, x, and one constant, a.

Example

Confirm Part 2 of the Fundamental Theorem of Calculus.

$$F(t) = \int_2^x t^3 + 2t^4\ dt$$

When we integrate we get

$$F(t) = \left(\frac{t^4}{4} + \frac{2t^5}{5} + C\right)\Bigg|_2^x$$

Evaluating over the interval, we get

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\left[\frac{(2)^4}{4}+\frac{2(2)^5}{5}+C\right]$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\left(4+\frac{64}{5}+C\right)$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\left(\frac{20}{5}+\frac{64}{5}+C\right)$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}+C-\frac{84}{5}-C$$

$$F(x)=\frac{x^4}{4}+\frac{2x^5}{5}-\frac{84}{5}$$

For the final step, we need to take the derivative of $F(x)$.

$$\frac{dF(x)}{dx}=\frac{d\left(\frac{x^4}{4}+\frac{2x^5}{5}-\frac{84}{5}\right)}{dx}$$

$$\frac{dF(x)}{dx}=\left(\frac{4x^3}{4}+\frac{10x^4}{5}-0\right)$$

$$\frac{dF(x)}{dx}=x^3+2x^4$$

We know that

$$f(t)=t^3+2t^4$$

So by substituting x for t we get

$$f(x)=x^3+2x^4$$

We can see that $\frac{dF(x)}{dx} = f(x)$.

$$\frac{dF(x)}{dx} = x^3 + 2x^4 = f(x)$$

Our final answer confirms Part 2 of the FTC.

Initial value problems

Consider the following situation. You're given the function $f(x) = 2x - 3$ and asked to find its derivative. This function is pretty basic, so unless you're taking calculus out of order, it shouldn't cause you too much stress to figure out that the derivative of $f(x)$ is 2.

Now consider what it would be like to work backwards from our derivative. If you're given the function $f'(x) = 2$ and asked to find its integral, it's impossible for you to get back to the original function, $f(x) = 2x - 3$. As you can see, taking the integral of the derivative we found gives us back the first term of the original function, $2x$, but somewhere along the way we lost the -3. In fact, we always lose the constant (term without a variable attached), when we take the derivative of something. Which means we're never going to get the constant back when we try to integrate our derivative. It's lost forever.

Accounting for that lost constant is why we always add C to the end of our integrals. C is called the "constant of integration" and it acts as a placeholder for our missing constant. In order to get back to our original function, and find our long-lost friend, -3 , we'll need some additional information about this problem, namely, an initial condition, which looks like this:

$$y(0) = -3$$

Problems that provide you with one or more initial conditions are called Initial Value Problems. Initial conditions take what would otherwise be an entire rainbow of possible solutions, and whittles them down to one specific solution.

Remember that the basic idea behind Initial Value Problems is that, once you differentiate a function, you lose some information about that function. More specifically, you lose the constant. By integrating $f'(x)$, you get a family of solutions that only differ by a constant.

$$\int 2\ dx = 2x - 3$$

$$\int 2\ dx = 2x + 7$$

$$\int 2\ dx = 2x - \sqrt{2}$$

Given one point on the function, (the initial condition), you can pick a specific solution out of a much broader solution set.

Example

Given $f'(x) = 2$ and $f(0) = -3$, find $f(x)$.

Integrating $f'(x)$ means we're integrating $2\ dx$, and we'll get $2x + C$, where C is the constant of integration. At this point, C is holding the place of our now familiar friend, -3, but we don't know that yet. We have to use our initial condition to find out.

To use our initial condition, $f(0) = -3$, we plug in the number inside the parentheses for x and the number on the right side of the equation for y. Therefore, in our case, we'll plug in 0 for x and -3 for y.

$$-3 = 2(0) + C$$

$$-3 = C$$

Notice that the solution would have been different had we been given a different initial condition. Now we know exactly what the full solution looks like, and exactly which one of the many possible solutions was originally differentiated. Therefore, the final answer is the function we originally differentiated:

$$f(x) = 2x - 3$$

Solving Integrals 5

Now that we know what an integral is, we'll talk about different techniques we can use to solve integrals.

U-substitution

Finding derivatives of elementary functions was a relatively simple process, because taking the derivative only meant applying the right derivative rules.

This is not the case with integration. Unlike derivatives, it may not be immediately clear which integration rules to use, and every function is like a puzzle.

Most integrals need some work before you can even begin the integration. They have to be transformed or manipulated in order to reduce the function's form into some simpler form. U-substitution is the simplest tool we have to transform integrals.

When you use u-substitution, you'll define u as a differentiable function in terms of the variable in the integral, take the derivative of u to get du, and then substitute these values back into your integrals.

Unfortunately, there are no perfect rules for defining u. If you try a substitution that doesn't work, just try another one. With practice, you'll get faster at identifying the right value for u.

Here are some common substitutions you can try.

For integrals that contain power functions, try using the base of the power function as the substitution.

Example

Use u-substitution to evaluate the integral.

$$\int x\left(x^2+1\right)^4\ dx$$

Let

$$u=x^2+1$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

Substituting back into the integral, we get

$$\int x(u)^4\ \frac{du}{2x}$$

$$\int u^4\ \frac{du}{2}$$

$$\frac{1}{2}\int u^4\ du$$

This is much simpler than our original integral, and something we can actually integrate.

$$\frac{1}{2}\left(\frac{1}{5}u^5\right)+C$$

$$\frac{1}{10}u^5+C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{10}\left(x^2+1\right)^5+C$$

For integrals of rational functions, if the numerator is of equal or greater degree than the denominator, always perform division first. Otherwise, try using the denominator as a possible substitution.

Example

Use u-substitution to evaluate the integral.

$$\int \frac{x}{x^2+1}\ dx$$

Let

$$u=x^2+1$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

Substituting back into the integral, we get

$$\int \frac{x}{u}\cdot\frac{du}{2x}$$

$$\int \frac{1}{u} \cdot \frac{du}{2}$$

$$\frac{1}{2} \int \frac{1}{u} \ du$$

This is much simpler than our original integral, and something we can actually integrate.

$$\frac{1}{2} \ln |u| + C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{2} \ln \left| x^2 + 1 \right| + C$$

For integrals containing exponential functions, try using the power for the substitution.

Example

Use u-substitution to evaluate the integral.

$$\int e^{\sin x \cos x} \cos 2x \ dx$$

Let $u = \sin x \cos x$, and using the product rule to differentiate,

$$du=\left[\left(\frac{d}{dx}\sin x\right)\cos x+\sin x\left(\frac{d}{dx}\cos x\right)\right]\ dx$$

$$du=\left[\cos x\cdot\cos x+\sin x\cdot(-\sin x)\right]\ dx$$

$$du=\cos^2 x-\sin^2 x\ dx$$

$$du=\cos 2x\ dx$$

Substituting back into the integral, we get

$$\int e^u\ du$$

$$e^u+C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$e^{\sin x\cos x}+C$$

Integrals containing trigonometric functions can be more challenging to manipulate. Sometimes, the value of u isn't even part of the original integral. Therefore, the better you know your trigonometric identities, the better off you'll be.

Example

Use u-substitution to evaluate the integral.

$$\int\frac{\tan x}{\cos x}\ dx$$

Since

$$\tan x = \frac{\sin x}{\cos x}$$

we can rewrite the integral as

$$\int \frac{\frac{\sin x}{\cos x}}{\cos x}\ dx$$

$$\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\ dx$$

$$\int \frac{\sin x}{\cos^2 x}\ dx$$

Let

$$u = \cos x$$

$$du = -\sin x\ dx$$

$$dx = -\frac{du}{\sin x}$$

Substituting back into the integral, we get

$$\int \frac{\sin x}{u^2} \cdot \left(-\frac{du}{\sin x}\right)$$

$$-\int \frac{1}{u^2}\ du$$

$$-\int u^{-2}\ du$$

$$-\frac{1}{-1}u^{-1}+C$$

$$u^{-1}+C$$

$$\frac{1}{u}+C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{\cos x}+C$$

Integration by parts

Unlike differentiation, integration is not always straightforward and we can't always express the integral of every function in terms of neat and clean elementary functions.

When your integral is too complicated to solve without a fancy technique and you've ruled out u-substitution, integration by parts should be your next approach for evaluating your integral. If you remember that the product rule was your method for finding derivatives of functions that were multiplied together, you can think about integration by parts as the method often used for integrating functions that are multiplied together.

Suppose you want to integrate the following

$$\int xe^{-x}\ dx$$

How can you integrate the above expression quickly and easily? You can't, unless you're a super human genius. But hopefully you can recognize that you have two functions multiplied together inside of this integral, one being x and the other being e^{-x}. If you try u-substitution, you won't find anything to cancel in your integral, and you'll be no better off, which means that your next step should be an attempt at integrating with our new method, integration by parts.

The formula we'll use is derived by integrating the product rule from, and looks like this:

$$\int u\ dv = uv - \int v\ du$$

In the formula above, everything to the left of the equals sign represents your original function, which means your original function must be composed of u and dv. Your job is to identify which part of your original function will be u, and which will be dv.

My favorite technique for picking u and dv is to assign u to the function in your integral whose derivative is simpler than the original u. Consider again the example from earlier:

$$\int xe^{-x}\ dx$$

I would assign u to x, because the derivative of x is 1, which is much simpler than x. If you have $\ln x$ in your integral, that's usually a good bet for u because the derivative of $\ln x$ is $1/x$; much simpler than $\ln x$. Once you pick which of your functions will be represented by u, the rest is easy because you know that the other function will be represented by dv.

Using this formula can be challenging for a lot of students, but the hardest part is identifying which of your two functions will be u and which will be dv. That's the very first thing you have to tackle with integration by parts, so once you get that over with, you'll be home free.

After completing this first crucial step, you take the derivative of u, called du, and the integral of dv, which will be v. Now that you have u, du, v and dv, you can plug all of your components into the right side of the integration by parts formula. Everything to the right of the equals sign will be part of your answer. If you've correctly assigned u and dv, the integral on the right should now be much easier to integrate.

Example

Evaluate the integral.

$$\int xe^{-x}\ dx$$

Our integral is comprised of two functions, x and e^{-x}. One of them must be u and the other dv. Since the derivative of x is 1, which is much simpler than the derivative of e^{-x}, we'll assign u to x.

$u = x$	→ differentiate →	$du = 1\ dx$
$dv = e^{-x}\ dx$	→ integrate →	$v = -e^{-x}$

Plugging all four components into the right side of our formula gives the following transformation of our original function:

$$(x)(-e^{-x}) - \int (-e^{-x})(1\ dx)$$

$$-xe^{-x} + \int e^{-x}\ dx$$

Now that we have something we can work with, we integrate.

$$-xe^{-x} + (-e^{-x}) + C$$

The answer is therefore

$$-xe^{-x} - e^{-x} + C$$

Or factored, we have

$$-e^{-x}(x+1)+C$$

Partial fractions

The method of partial fractions is an extremely useful tool whenever you need to integrate a fraction with polynomials in both the numerator and denominator; something like this:

$$f(x)=\frac{7x+1}{x^2-1}$$

If you were asked to integrate

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

you shouldn't have too much trouble, because if you don't have a variable in the numerator of your fraction, then your integral is simply the numerator multiplied by the natural log ($\ln$) of the absolute value of the denominator, like this:

$$\int \frac{3}{x+1}+\frac{4}{x-1}\ dx$$

$$3\ln|x+1|+4\ln|x-1|+C$$

where C is the constant of integration. Not too hard, right?

Don't forget to use chain rule and divide by the derivative of your denominator. In the case above, the derivatives of both of our denominators are 1, so this step didn't appear. But if your integral is

$$\int \frac{3}{2x+1}\ dx$$

then your answer will be

$$\frac{3}{2}\ln|2x+1|+C$$

because the derivative of our denominator is 2, which means we have to divide by 2, according to chain rule.

So back to the original example. We said at the beginning of this section that

$$f(x)=\frac{7x+1}{x^2-1}$$

would be difficult to integrate, but that we wouldn't have as much trouble with

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

In fact, these two are actually the same function. If we try adding $3/(x+1)$ and $4/(x-1)$ together, you'll see that we get back to the original function.

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

$$f(x)=\frac{3(x-1)+4(x+1)}{(x+1)(x-1)}$$

$$f(x)=\frac{3x-3+4x+4}{x^2-x+x-1}$$

$$f(x)=\frac{7x+1}{x^2-1}$$

Again, attempting to integrate $f(x) = (7x + 1)/\left(x^2 - 1\right)$ is extremely difficult. But if you can express this function as $f(x) = 3/(x + 1) + 4/(x - 1)$, then integrating is much simpler. This method of converting complicated fractions into simpler fractions that are easier to integrate is called decomposition into "partial fractions".

Let's start talking about how to perform a partial fractions decomposition. Before we move forward it's important to remember that you must perform long division with your polynomials whenever the degree (value of the greatest exponent) of your denominator is not greater than the degree of your numerator, as is the case in the following example.

Example

Evaluate the integral.

$$\int \frac{x^3 - 3x^2 + 2}{x + 3}\ dx$$

Because the degree (the value of the highest exponent in the numerator, 3), is greater than the degree of the denominator, 1, we have to perform long division first.

$$x^2 - 6x + 18 - \frac{52}{x + 3}$$

$$x + 3 \qquad x^3 - 3x^2 + 0x + 2$$

$$-\left(x^3 + 3x^2\right)$$

$$-6x^2$$

$$-\left(-6x^2 - 18x\right)$$

$$18x + 2$$

$$-\left(18x + 54\right)$$

$$-52$$

After performing long division, our fraction has been decomposed into

$$\left(x^2 - 6x + 18\right) - \frac{52}{x+3}$$

Now the function is easy to integrate.

$$\int x^2 - 6x + 18 - \frac{52}{x+3}\ dx$$

$$\frac{1}{3}x^3 - 3x^2 + 18x - 52\ln|x+3| + C$$

Okay. So now that you've either performed long division or confirmed that the degree of your denominator is greater than the degree of your numerator (such that you don't have to perform long division), it's time for full-blown partial fractions. Oh goodie! I hope you're excited.

The first step is to factor your denominator as much as you can. Your second step will be determining which type of denominator you're dealing with, depending on how it factors. Your denominator will be the product of the following:

1. Distinct linear factors

2. Repeated linear factors

3. Distinct quadratic factors

4. Repeated quadratic factors

Let's take a look at an example of each of these four cases so that you understand the difference between them.

Distinct linear factors

In this first example, we'll look at the first case above, in which the denominator is a product of distinct linear factors.

Example

Evaluate the integral.

$$\int \frac{x^2+2x+1}{x^3-2x^2-x+2}\ dx$$

Since the degree of the denominator is higher than the degree of the numerator, we don't have to perform long division before we start. Instead, we can move straight to factoring the denominator, as follows.

$$\int \frac{x^2+2x+1}{(x-1)(x+1)(x-2)}\ dx$$

We can see that our denominator is a product of distinct linear factors because $(x-1)$, $(x+1)$, and $(x-2)$ are all different first-degree factors.

Once we have it factored, we set our fraction equal to the sum of its component parts, assigning new variables to the numerator of each of our fractions. Since our denominator can be broken down into three different factors, we need three variables A, B and C to go on top of each one of our new fractions, like so:

$$\frac{x^2+2x+1}{(x-1)(x+1)(x-2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}$$

Now that we've separated our original function into its partial fractions, we multiply both sides by the denominator of the left-hand side. The denominator will cancel on the left-hand side, and on the right, each of the three partial fractions will end up multiplied by all the factors other than the one that was previously included in its denominator.

$$x^2+2x+1=A(x+1)(x-2)+B(x-1)(x-2)+C(x-1)(x+1)$$

The next step is to multiply out all of these terms.

$$x^2+2x+1=A\left(x^2-x-2\right)+B\left(x^2-3x+2\right)+C\left(x^2-1\right)$$

$$x^2+2x+1=Ax^2-Ax-2A+Bx^2-3Bx+2B+Cx^2-C$$

Now we collect like terms together, meaning that we re-order them, putting all the x^2 terms next to each other, all the x terms next to each other, and then all the constants next to each other.

$$x^2+2x+1=\left(Ax^2+Bx^2+Cx^2\right)+(-Ax-3Bx)+(-2A+2B-C)$$

Finally, we factor out the x terms.

$$x^2 + 2x + 1 = (A + B + C)x^2 + (-A - 3B)x + (-2A + 2B - C)$$

Doing this allows us to equate coefficients on the left and right sides. Do you see how the coefficient on the x^2 term on the left-hand side of the equation is 1? Well, the coefficient on the x^2 term on the right-hand side is $(A + B + C)$, which means those two must be equal. We can do the same for the x term, as well as for the constants. We get the following three equations:

[1] $A + B + C = 1$

[2] $-A - 3B = 2$

[3] $-2A + 2B - C = 1$

Now that we have these equations, we need to solve for our three constants A, B, and C. This can easily get confusing, but with practice, you should get the hang of it. If you have one equation with only two variables instead of all three, like [2], that's a good place to start. Solving [2] for A gives us

[4] $A = -3B - 2$

Now we'll substitute [4] for A into [1] and [3] and then simplify, such that these equations:

$$(-3B - 2) + B + C = 1$$

$$-2(-3B - 2) + 2B - C = 1$$

become these equations:

[5] $-2B + C = 3$

[6] $8B - C = -3$

Now we can add [5] and [6] together to solve for B.

$$-2B + C + 8B - C = 3 - 3$$

$$6B = 0$$

[7] $B = 0$

Plugging [7] back into [4] to find A, we get

$$A = -3(0) - 2$$

[8] $A = -2$

Plugging [7] back into [5] to find B, we get

$$-2(0) + C = 3$$

[9] $C = 3$

Having solved for the values of our three constants in [7], [8] and [9], we're finally ready to plug them back into our partial fractions decomposition. Doing so should produce something that's easier for us to integrate than our original function.

$$\int \frac{x^2 + 2x + 1}{(x-1)(x+1)(x-2)}\ dx = \int \frac{-2}{x-1} + \frac{0}{x+1} + \frac{3}{x-2}\ dx$$

Simplifying the integral on the right side, we get

$$\int \frac{3}{x-2} - \frac{2}{x-1}\ dx$$

Remembering that the integral of $1/x$ is $\ln|x| + C$, we integrate and get

$$3\ln|x-2| - 2\ln|x-1| + C$$

And using laws of logarithms to simplify the final answer, we get

$$\frac{3}{2}\ln\left|\frac{x-2}{x-1}\right| + C$$

Repeated linear factors

Let's move now to the second of our four case types above, in which the denominator will be a product of linear factors, some of which are repeated.

Example

Evaluate the integral.

$$\int \frac{2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13}{x^4 - 2x^2 + 1}\ dx$$

You'll see that we need to carry out long division before we start factoring, since the degree of the numerator is greater than the degree of the denominator ($5 > 4$).

$$2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1}$$

$$x^4 - 2x^2 + 1 \qquad 2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13$$

$$-\left(2x^5 + 0x^4 - 4x^3 + 0x^2 + 2x\right)$$

$$-3x^4 + 9x^3 + 3x^2 - 11x + 13$$

$$-\left(-3x^4 + 0x^3 + 6x^2 + 0x - 3\right)$$

$$9x^3 - 3x^2 - 11x + 16$$

Now that the degree of the remainder is less than the degree of the original denominator, we can rewrite the problem as

$$\int 2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1}\ dx$$

Integrating the $2x - 3$ will be simple, so for now, let's focus on the fraction. We'll factor the denominator.

$$\frac{9x^3 - 3x^2 - 11x + 16}{\left(x^2 - 1\right)\left(x^2 - 1\right)}$$

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x + 1)(x - 1)(x + 1)(x - 1)}$$

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x + 1)^2(x - 1)^2}$$

Given the factors involved in our denominator, you might think that the partial fraction decomposition would look like this:

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x + 1)^2(x - 1)^2} = \frac{A}{x + 1} + \frac{B}{x + 1} + \frac{C}{x - 1} + \frac{D}{x - 1}$$

However, the fact that we're dealing with repeated factors, ($(x+1)$ is a factor twice and $(x-1)$ is a factor twice), the partial fractions decomposition is actually the following:

$$\frac{9x^3-3x^2-11x+16}{x^4-2x^2+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

To see why, let's take a simpler example. The partial fractions decomposition of $x^2/\left[(x+1)^4\right]$ is

$$\frac{x^2}{(x+1)^4}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{(x+1)^4}$$

Notice how we included $(x+1)^4$, our original factor, as well as each factor of lesser degree? We have to do this every time we have a repeated factor.

Let's continue with our original example.

$$\frac{9x^3-3x^2-11x+16}{x^4-2x^2+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

We'll multiply both sides of our equation by the denominator from the left side, $(x+1)^2(x-1)^2$, which will cancel the denominator on the left and some of the factors on the right.

$$9x^3-3x^2-11x+16=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

To simplify, we'll start multiplying all terms on the right side together.

$$9x^3-3x^2-11x+16=A\left(x^3+x^2-x-1\right)+B\left(x^2+2x+1\right)$$

$$+C\left(x^3-x^2-x+1\right)+D\left(x^2-2x+1\right)$$

Now we'll group like terms together.

$$9x^3 - 3x^2 - 11x + 16 = (A + C)x^3 + (A + B - C + D)x^2$$

$$+(-A + 2B - C - 2D)x + (-A + B + C + D)$$

Equating coefficients on both sides of the equation gives us the following equations.

[1] $A + C = 9$

[2] $A + B - C + D = -3$

[3] $-A + 2B - C - 2D = -11$

[4] $-A + B + C + D = 16$

Now we'll start solving for variables. If we subtract A from both sides of [1], we get

[5] $C = 9 - A$

If we plug [5] into [2], [3] and [4], we have

$$A + B - (9 - A) + D = -3$$

$$-A + 2B - (9 - A) - 2D = -11$$

$$-A + B + (9 - A) + D = 16$$

And simplifying, we get the following:

[6] $2A + B + D = 6$

[7] $2B - 2D = -2$

[8] $-2A+B+D=7$

Let's now solve **[7]** for B.

$$2B-2D=-2$$

$$2B=-2+2D$$

$$B=-1+D$$

[9] $B=D-1$

Plugging **[9]** into **[6]** and **[8]**, we get

$$2A+(D-1)+D=6$$

$$-2A+(D-1)+D=7$$

And simplifying, we get the following:

[10] $2A+2D=7$

[11] $-2A+2D=8$

We solve **[11]** for D.

$$-2A+2D=8$$

$$2D=8+2A$$

[12] $D=4+A$

We plug **[12]** into **[10]** to solve for A.

$$2A+2(4+A)=7$$

$$2A+8+2A=7$$

$$4A = -1$$

[13] $$A = -\frac{1}{4}$$

At last! We've solved for one variable. Now it's pretty quick to find the other three. With [13], we can use [12] to find D.

$$D = 4 - \frac{1}{4}$$

[14] $$D = \frac{15}{4}$$

We plug [14] into [9] to find B.

$$B = \frac{15}{4} - 1$$

[15] $$B = \frac{11}{4}$$

Last but not least, we plug [13] into [5] to solve for C.

$$C = 9 - \left(-\frac{1}{4}\right)$$

$$C = 9 + \frac{1}{4}$$

[16] $$C = \frac{37}{4}$$

Taking the values of the constants from [13], [14], [15], [16] and bringing back the $2x - 3$ that we put aside following the long division earlier in this example, we'll write out the partial fractions decomposition.

$$\int \frac{2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13}{x^4 - 2x^2 + 1} \ dx$$

$$\int 2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} \ dx$$

$$\int 2x - 3 + \frac{-\frac{1}{4}}{x - 1} + \frac{\frac{11}{4}}{(x - 1)^2} + \frac{\frac{37}{4}}{x + 1} + \frac{\frac{15}{4}}{(x + 1)^2} \ dx$$

Now we can integrate. Using the rule from algebra that $1/(x^n) = x^{-n}$, we'll flip the second and fourth fractions so that they are easier to integrate.

$$\int 2x - 3 \ dx - \frac{1}{4}\int \frac{1}{x - 1} \ dx + \frac{11}{4}\int (x - 1)^{-2} \ dx + \frac{37}{4}\int \frac{1}{x + 1} \ dx + \frac{15}{4}\int (x + 1)^{-2} \ dx$$

Now that we've simplified, we'll integrate to get our final answer.

$$x^2 - 3x - \frac{1}{4}\ln|x - 1| - \frac{11}{4(x - 1)} + \frac{37}{4}\ln|x + 1| - \frac{15}{4(x + 1)} + C$$

Distinct quadratic factors

Now let's take a look at an example in which the denominator is a product of distinct quadratic factors.

In order to solve these types of integrals, you'll sometimes need the following formula:

[A] $$\int \frac{1}{mx^2 + n^2} \ dx = \frac{m}{n} \tan^{-1}\left(\frac{x}{n}\right) + C$$

Example

Evaluate the integral.

$$\int \frac{x^2 - 2x - 5}{x^3 - x^2 + 9x - 9}\ dx$$

As always, the first thing to notice is that the degree of the denominator is larger than the degree of the numerator, which means that we don't have to perform long division before we can start factoring the denominator. So let's get right to it and factor the denominator.

$$\int \frac{x^2 - 2x - 5}{(x - 1)\left(x^2 + 9\right)}\ dx$$

We have one distinct linear factor, $(x - 1)$, and one distinct quadratic factor, $\left(x^2 + 9\right)$.

As we already know, linear factors require one constant in the numerator, like this:

$$\frac{A}{x - 1}$$

The numerators of quadratic factors require a polynomial, like this:

$$\frac{Ax + B}{x^2 + 9}$$

Remember though that when we add these fractions together in the partial fractions decomposition, we never want to repeat the same constant, so the partial fractions decomposition is

$$\frac{x^2-2x-5}{(x-1)\left(x^2+9\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9}$$

See how we started the second fraction with B instead of A? If we added a second quadratic factor to this example, its numerator would be $Dx+E$.

Multiplying both sides of our decomposition by the denominator on the left gives

$$x^2-2x-5=A\left(x^2+9\right)+(Bx+C)(x-1)$$

$$x^2-2x-5=Ax^2+9A+Bx^2-Bx+Cx-C$$

$$x^2-2x-5=\left(Ax^2+Bx^2\right)+(-Bx+Cx)+(9A-C)$$

$$x^2-2x-5=(A+B)x^2+(-B+C)x+(9A-C)$$

Then equating coefficients on the left and right sides gives us the following equations.

[1] $A+B=1$

[2] $-B+C=-2$

[3] $9A-C=-5$

We solve [1] for A.

[4] $A=1-B$

Plugging [4] into [3] leaves us with two equations in terms of B and C.

[2] $-B+C=-2$

[5] $9(1-B)-C=-5$

Simplifying [5] leaves us with

[2] $-B + C = -2$

[6] $-9B - C = -14$

Solving [2] for C we get

[7] $C = B - 2$

Plugging [7] into [6] gives

$$-9B - (B - 2) = -14$$

$$-10B + 2 = -14$$

$$-10B = -16$$

[8] $B = \frac{8}{5}$

Now that we have a value for B, we'll plug [8] into [7] to solve for C.

$$C = \frac{8}{5} - 2$$

[9] $C = -\frac{2}{5}$

We can also plug [8] into [4] to solve for A.

$$A = 1 - \frac{8}{5}$$

[10] $A = -\frac{3}{5}$

Plugging [8], [9] and [10] into our partial fractions decomposition, we get

$$\int \frac{x^2 - 2x - 5}{(x-1)(x^2+9)}\ dx = \int \frac{-\frac{3}{5}}{x-1} + \frac{\frac{8}{5}x - \frac{2}{5}}{x^2+9}\ dx$$

$$-\frac{3}{5}\int \frac{1}{x-1}\ dx + \frac{8}{5}\int \frac{x}{x^2+9}\ dx - \frac{2}{5}\int \frac{1}{x^2+9}\ dx$$

Integrating the first term only, we get

$$-\frac{3}{5}\ln|x-1| + \frac{8}{5}\int \frac{x}{x^2+9}\ dx - \frac{2}{5}\int \frac{1}{x^2+9}\ dx$$

Using u-substitution to integrate the second integral, letting

$$u = x^2 + 9$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$-\frac{3}{5}\ln|x-1| + \frac{8}{5}\int \frac{x}{u} \cdot \frac{du}{2x} - \frac{2}{5}\int \frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\int \frac{1}{u}\ du - \frac{2}{5}\int \frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\ln|u| - \frac{2}{5}\int \frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\ln\left|x^2+9\right| - \frac{2}{5}\int \frac{1}{x^2+9}\ dx$$

Using [A] to integrate the third term, we get

[A] $\displaystyle\int \frac{1}{mx^2+n^2}\ dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right)+C$

$m=1$

$n=3$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\ln\left|x^2+9\right|-\frac{2}{5}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]+C$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\ln\left|x^2+9\right|-\frac{2}{15}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$$\frac{1}{5}\left[4\ln\left|x^2+9\right|-3\ln|x-1|-\frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]+C$$

Repeated quadratic factors

Last but not least, let's take a look at an example in which the denominator is a product of quadratic factors, at least some of which are repeated.

We'll be using formula [A] like we did in the last example.

Example

Evaluate the integral.

$$\int \frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}\ dx$$

Remember, when we're dealing with repeated factors, we have to include every lesser degree of that factor in our partial fractions decomposition, which will be

$$\frac{-x^3+2x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

Multiplying both sides by the denominator of the left-hand side gives us

$$-x^3+2x^2-x+1=A\left(x^2+1\right)^2+(Bx+C)x\left(x^2+1\right)+(Dx+E)x$$

Simplifying the right-hand side, we get

$$-x^3+2x^2-x+1=A\left(x^4+2x^2+1\right)+(Bx+C)\left(x^3+x\right)+(Dx+E)x$$

$$-x^3+2x^2-x+1=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

Grouping like terms together, we have

$$-x^3+2x^2-x+1=\left(Ax^4+Bx^4\right)+\left(Cx^3\right)+\left(2Ax^2+Bx^2+Dx^2\right)+(Cx+Ex)+(A)$$

And factoring, we get

$$-x^3+2x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+(A)$$

Now we equate coefficients and write down the equations we'll use to solve for each of our constants.

[1] $A+B=0$

[2] $C=-1$

[3] $2A+B+D=2$

[4] $C + E = -1$

[5] $A = 1$

We already have values for A and C. Plugging [5] into [1] to solve for B gives us

$$1 + B = 0$$

[6] $B = -1$

Plugging [2] into [4] to solve for E, we get

$$-1 + E = -1$$

[7] $E = 0$

Plugging [5] and [6] into [3] to solve for D gives us

$$2(1) - 1 + D = 2$$

[8] $D = 1$

Plugging our constants from [2], [5], [6], [7] and [8] back into the decomposition, we get

$$\int \frac{(1)}{x} + \frac{(-1)x + (-1)}{x^2 + 1} + \frac{(1)x + (0)}{\left(x^2 + 1\right)^2}\ dx$$

$$\int \frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{\left(x^2 + 1\right)^2}\ dx$$

$$\int \frac{1}{x}\ dx - \int \frac{x + 1}{x^2 + 1}\ dx + \int \frac{x}{\left(x^2 + 1\right)^2}\ dx$$

$$\int \frac{1}{x}\ dx - \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

Evaluating the first integral only, we get

$$\ln|x| - \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

Using u-substitution to evaluate the second integral, letting

$$u = x^2 + 1$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$\ln|x| - \int \frac{x}{u} \cdot \frac{du}{2x} - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\int \frac{1}{u}\ du - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|u| - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

Using formula [A] to evaluate the third integral, we get

[A] $$\int \frac{1}{mx^2+n^2}\ dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right)+C$$

$$m = 1$$

$$n = 1$$

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right) + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}x + \int \frac{x}{\left(x^2+1\right)^2}\ dx$$

Using u-substitution to evaluate the fourth integral, letting

$$u = x^2+1$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}x + \int \frac{x}{u^2}\cdot\frac{du}{2x}$$

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}x + \frac{1}{2}\int \frac{1}{u^2}\ du$$

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}x + \frac{1}{2}\int u^{-2}\ du$$

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}x - \frac{1}{2u} + C$$

And plugging back in for u gives us the final answer.

$$\ln|x| - \frac{1}{2}\ln\left|x^2+1\right| - \tan^{-1}x - \frac{1}{2\left(x^2+1\right)} + C$$

In summary, in order to integrate by expressing rational functions (fractions) in terms of their partial fractions decomposition, you should follow these steps:

1. Ensure that the rational function is “proper”, such that the degree (greatest exponent) of the numerator is less than the degree of the denominator. If necessary, use long division to make it proper.

2. Perform the partial fractions decomposition by factoring the denominator, which will always be expressible as the product of either linear or quadratic factors, some of which may be repeated.

    a. If the denominator is a product of distinct linear factors: This is the simplest kind of partial fractions decomposition. Nothing fancy here.

    b. If the denominator is a product of linear factors, some of which are repeated: Remember to include factors of lesser degree than your repeated factors.

    c. If the denominator is a product of distinct quadratic factors: You'll need the following equation:

[A] $$\int \frac{1}{mx^2+n^2}\,dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right) + C$$

d. If the denominator is a product of quadratic factors, some of which are repeated: Use the two formulas above and remember to include factors of lesser degree than your repeated factors.

Partial Derivatives 6

Let's expand our knowledge of derivatives to multivariable functions, where we'll learn that we'll need one derivative per variable in order to describe the derivative of a multivariable function.

Partial derivatives in two variables

By this point we've already learned how to find derivatives of single-variable functions. After learning derivative rules like power rule, product rule, quotient rule, chain rule and others, we're pretty comfortable handling the derivatives of functions like these:

$$f(x) = x^2 + 5$$

$$f(x) = \frac{\left(x^2 + 4\right)^3 \sin x}{x^4 + \ln 7x^4}$$

But now it's time to start talking about derivatives of multivariable functions, such as

$$f(x, y) = x^4y^3 + x^3y^2 + \ln xe^y$$

Finding derivatives of a multivariable function like this one may be less challenging than you think, because we're actually only going to take the derivative with respect to one variable at a time. For example, we'll take the derivative with respect to x while we treat y like it's a constant. Then we'll take another derivative of the original function, this one with respect to y, and we'll treat x as a constant.

In that way, we sort of reduce the problem to a single-variable derivative problem, which is a derivative we already know how to handle!

We call these kinds of derivatives "partial derivatives" because we're only taking the derivative of one part (variable) of the function at a time.

Remember the definition of the derivative from single-variable calculus (aka the difference quotient)? Let's adapt that definition so that it works for us for multivariable functions.

We know that, if z is a function defined in terms of x and y, like $z = f(x, y)$, then

The partial derivative of z with respect to x is

$$z_x = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The partial derivative of z with respect to y is

$$z_y = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

The definition as we've written it here gives two different kinds of notation for the partial derivatives of z: z_x or z_y and $f_x(x, y)$ or $f_y(x, y)$. In fact, there are many ways you might see partial derivatives defined.

The partial derivatives of a function z defined in terms of x and y could be written in all of these ways:

The partial derivative of z with respect to x:

$$f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x = z_x$$

The partial derivative of z with respect to y:

$$f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y = z_y$$

Let's use what we've learned so far to work through an example using the difference quotient to find the partial derivatives of a multivariable function.

Example

Using the definition, find the partial derivatives of

$$f(x, y) = 2x^2y$$

For the partial derivative of z with respect to x, we'll substitute $x + h$ into the original function for x.

$$f(x + h, y) = 2(x + h)^2y$$

$$f(x + h, y) = 2\left(x^2 + 2xh + h^2\right)y$$

$$f(x + h, y) = 2x^2y + 4xhy + 2h^2y$$

Plugging our values of $f(x, y)$ and $f(x + h, y)$ into the definition, we get

$$f_x(x, y) = \lim_{h\to 0} \frac{2x^2y + 4xhy + 2h^2y - 2x^2y}{h}$$

$$f_x(x, y) = \lim_{h\to 0} \frac{4xhy + 2h^2y}{h}$$

$$f_x(x, y) = \lim_{h\to 0} 4xy + 2hy$$

$$f_x(x, y) = \lim_{h\to 0} 4xy + 2(0)y$$

$$f_x(x, y) = 4xy$$

For the partial derivative of z with respect to y, we'll substitute $y + h$ into the original function for y.

$$f(x, y + h) = 2x^2(y + h)$$

$$f(x, y + h) = 2x^2y + 2x^2h$$

Plugging our values of $f(x, y)$ and $f(x, y + h)$ into the definition, we get

$$f_y(x, y) = \lim_{h \to 0} \frac{2x^2y + 2x^2h - 2x^2y}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{2x^2h}{h}$$

$$f_y(x, y) = \lim_{h \to 0} 2x^2$$

$$f_y(x, y) = 2x^2$$

You'll remember from single-variable calculus that using the definition of the derivative was the "long way" that we learned to take the derivative before we learned the derivative rules that made the process faster. The good news is that we can apply all the same derivative rules to multivariable functions to avoid using the difference quotient! We just have to remember to work with only one variable at a time, treating all other variables as constants.

The next example shows how the power rule provides a faster way to find this function's partial derivatives.

Example

Using the power rule, find the partial derivatives of

$$f(x, y) = 2x^2y$$

For the partial derivative of z with respect to x, we treat y as a constant and use power rule to find the derivative.

$$f_x(x, y) = 2\left(\frac{d}{dx}x^2\right)y$$

$$f_x(x, y) = 2(2x)y$$

$$f_x(x, y) = 4xy$$

For the partial derivative of z with respect to y, we treat x as a constant and use power rule to find the derivative.

$$f_y(x, y) = 2x^2\left(\frac{d}{dy}y\right)$$

$$f_y(x, y) = 2x^2(1)$$

$$f_y(x, y) = 2x^2$$

Second-order partial derivatives

We already learned in single-variable calculus how to find second derivatives; we just took the derivative of the derivative. Remember how we even used the second derivative to help us with inflection points and concavity when we were learning optimization and sketching graphs?

Here's an example from single variable calculus of what a second derivative looks like:

$$f(x) = 2x^3$$

$$f'(x) = 6x^2$$

$$f''(x) = 12x$$

Well, we can find the second derivative of a multivariable function in the same way. Except, instead of just one function that defines the second derivative (like $f''(x) = 12x$ above), we'll need four functions that define the second derivative! Our second-order partial derivatives will be:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

The derivative with respect to x, of the first-order partial derivative with respect to x

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

The derivative with respect to y, of the first-order partial derivative with respect to y

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

The derivative with respect to y, of the first-order partial derivative with respect to x

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$ The derivative with respect to x, of the first-order partial derivative with respect to y

That wording is a little bit complicated. We can think about like the illustration below, where we start with the original function in the first row, take first derivatives in the second row, and then second derivatives in the third row.

$$\begin{tikzcd}[column sep=small]
 & & & f(x,y) \arrow[dl, no head] \arrow[dr, no head] & & & \\
 & & \frac{\partial f}{\partial x} \arrow[dll, no head] \arrow[dl, no head] & & \frac{\partial f}{\partial y} \arrow[dr, no head] \arrow[drr, no head] & & \\
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & & & & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}$$

The good news is that, even though this looks like four second-order partial derivatives, it's actually only three. That's because the two second-order partial derivatives in the middle of the third row will always come out to be the same.

Whether you start with the first-order partial derivative with respect to x, and then take the partial derivative of that with respect to y; or if you start with the first-order partial derivative with respect to y, and then take the partial derivative of that with respect to x; you'll get the same answer in both cases. Which means our tree actually looks like this:

$$f(x, y)$$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial^2 f}{\partial y^2}$$

Example

Find the second-order partial derivatives of the multivariable function.

$$f(x, y) = 2x^2y$$

We found the first-order partial derivatives of this function in a previous section, and they were

$$f_x(x, y) = 4xy$$

$$f_y(x, y) = 2x^2$$

The four second order partial derivatives are:

$$f_{xx} = \frac{\partial}{\partial x}(4xy) = 4y$$

$$f_{xy} = \frac{\partial}{\partial x}\left(2x^2\right) = 4x$$

$$f_{yx} = \frac{\partial}{\partial y}(4xy) = 4x$$

$$f_{yy} = \frac{\partial}{\partial y}\left(2x^2\right) = 0$$

Notice that the mixed second-order partial derivative is the same, regardless of whether you take the partial derivative first with respect to x and then y, or vice versa.

Differential Equations 7

Differential equations let us look at the rate of change of one variable, with respect to another variable.

Introduction to differential equations

Differential equations are broadly classified into two categories:

1. Partial Differential Equations (PDEs)

2. Ordinary Differential Equations (ODEs)

We discussed partial differential equations (partial derivatives) previously, so here we'll be discussing only ordinary differential equations.

ODEs involve the "ordinary" derivative of a function of a single variable, while PDEs involve partial derivatives of functions of multiple variables. So as we saw before,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is a partial derivative. In contrast, an ordinary differential equation looks like this:

$$\frac{d^2 y}{dx^2} = 2x$$

ODEs can be classified as

1. Linear differential equations

2. Nonlinear (separable) differential equations

ODEs can also be classified according to their order. When we talk about the “order” of a differential equation, we mean the derivative of the highest degree that occurs in the equation.

For example, the order of

$$\frac{dy}{dx} - \sin x \cos x = 2x$$

is 1, because the highest-degree derivative that occurs in this equation is dy/dx.

In contrast, the order of

$$\frac{d^2y}{dx^2} - 3 = \frac{dy}{dx}$$

is 2, because the highest-degree derivative that occurs in this equation is d^2y/dx^2.

Linear, first-order differential equations

Here we'll be discussing linear, first-order differential equations. Remember from the introduction to this section that these are ordinary differential equations (ODEs).

A linear, first-order differential equation will be expressed in the form

$$\text{[A]} \quad \frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x, the independent variable. Let's talk about how to solve a linear, first-order differential equation.

Example

Solve the differential equation.

$$x\frac{dy}{dx} - 2y = x^2$$

It's really important that the form of the differential equation match [A] exactly. In order to get dy/dx by itself in our equation, we need to divide both sides by x.

$$\text{[1]} \quad \frac{dy}{dx} - \frac{2}{x}y = x$$

Matching [1] to [A] above, we can see that

$$P(x) = -\frac{2}{x}$$

and

$$Q(x) = x$$

Once we're at a point where we've identified $P(x)$ and $Q(x)$ from the standard form of our linear, first-order differential equation, our next step is to identify our equation's "integrating factor". To find the integrating factor, we use the formula

[B] $$I(x) = e^{\int P(x)\, dx}$$

Since

$$P(x) = -\frac{2}{x}$$

the integrating factor for this equation is

$$I(x) = e^{\int -\frac{2}{x}\, dx}$$

$$I(x) = e^{-2\int \frac{1}{x}\, dx}$$

$$I(x) = e^{-2\ln x}$$

Note: You can leave out the constant of integration, C, when you integrate $P(x)$. You can take my word for it, or you can read through the very, very long proof that tells you why.

$$I(x) = e^{\ln x^{-2}}$$

$$I(x) = x^{-2}$$

Our integrating factor is

[2] $I(x) = \frac{1}{x^2}$

We'll multiply both sides of [1] by [2] to get

$$\frac{dy}{dx} \cdot \frac{1}{x^2} - \frac{2}{x} \cdot \frac{1}{x^2} \cdot y = x \cdot \frac{1}{x^2}$$

$$\frac{dy}{dx}\left(\frac{1}{x^2}\right) - \frac{2}{x^3}y = \frac{1}{x}$$

[3] $y'x^{-2} - 2yx^{-3} = x^{-1}$

The reason we multiply by the integrating factor is that it does something for us that's extremely convenient, even though we don't realize it yet.

It turns the left side of [3] is the derivative of yx^{-2}, in other words, y times our integrating factor, or $yI(x)$. And this will always be the case! Let's prove it by taking the derivative of yx^{-2}. We'll need to use product rule.

$$\frac{d}{dx}\left(yx^{-2}\right) = \left(\frac{d}{dx}y\right)\left(x^{-2}\right) + (y)\left(\frac{d}{dx}x^{-2}\right)$$

$$\frac{d}{dx}\left(yx^{-2}\right) = (y')\left(x^{-2}\right) + (y)\left(-2x^{-3}\right)$$

$$\frac{d}{dx}\left(yx^{-2}\right) = y'x^{-2} - 2yx^{-3}$$

See how the derivative we just found matches the left side of [3]? If we multiply through [1] by the integrating factor that we found, [2], the resulting left side will always be $\left[yI(x)\right]'$.

So we can substitute $\left[yx^{-2}\right]'$ into [3] to get

$$\left[yx^{-2}\right]' = x^{-1}$$

Now we integrate both sides.

$$\int \left[yx^{-2}\right]' \ dx = \int x^{-1} \ dx$$

$$yx^{-2} = \ln|x| + C$$

$$y = x^2 \left(\ln|x| + C\right)$$

Example

Solve the differential equation.

$$\frac{dy}{dx} + 2y = 4e^{-2x}$$

Our problem is already in standard form for a linear differential equation, so we can see that $P(x) = 2$ and $Q(x) = 4e^{-2x}$. We'll use $P(x)$ to find the integrating factor.

$$\rho(x) = e^{\int P(x) \ dx}$$

$$\rho(x) = e^{\int 2 \ dx}$$

$$\rho(x) = e^{2x}$$

Then we'll multiply through both sides of our linear differential equation by the integrating factor.

$$\frac{dy}{dx}\left(e^{2x}\right)+2y\left(e^{2x}\right)=4e^{-2x}\left(e^{2x}\right)$$

$$\frac{dy}{dx}e^{2x}+2e^{2x}y=4e^{0}$$

$$\frac{dy}{dx}e^{2x}+2e^{2x}y=4$$

To simplify the left-hand side further we need to remember the product rule for differentiation,

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

If we say that

$$f'(x)=\frac{dy}{dx}$$

and

$$g(x)=e^{2x}$$

then

$$f(x)=y$$

and

$$g'(x)=2e^{2x}$$

If we substitute all of that into the product rule formula, we get

$$\frac{d}{dx}\left(ye^{2x}\right) = \frac{dy}{dx}e^{2x} + y2e^{2x}$$

$$\frac{d}{dx}\left(ye^{2x}\right) = \frac{dy}{dx}e^{2x} + 2e^{2x}y$$

What we see now is that the right side of this equation matches exactly the left side of our linear differential equation after we multiplied through by the integrating factor. Therefore, we can make a substitution and replace the left side of our linear differential equation with the left side of the product rule formula.

$$\frac{d}{dx}\left(ye^{2x}\right) = 4$$

The goal is to a general solution for y. In order to take the next step to solve for y, we have to integrate both sides. Integrating the derivative d/dx will make both things cancel out.

$$\int \frac{d}{dx}\left(ye^{2x}\right)\ dx = \int 4\ dx$$

$$ye^{2x} = 4x + C$$

Dividing both sides by e^{2x} to get y by itself gives

$$y = \frac{4x + C}{e^{2x}}$$

This is the general solution to the linear differential equation.

Separable (nonlinear) differential equations

A separable, first-order differential equation is an equation in the following form

$$y' = f(x)g(y),$$

where $f(x)$ and $g(y)$ are functions of x and y, respectively. The dependent variable is y; the independent variable is x. We can easily integrate functions in this form by separating variables.

$$y' = f(x)g(y)$$

$$\frac{dy}{dx} = f(x)g(y)$$

$$dy = f(x)g(y)\ dx$$

$$\frac{dy}{g(y)} = f(x)\ dx$$

$$\frac{1}{g(y)}\ dy = f(x)\ dx$$

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx$$

Sometimes in our final answer, we'll be able to express y explicitly as a function of x, but not always. When we can't, we just have to be satisfied with an implicit function, where y and x are not cleanly separated by the $=$ sign.

Example

Solve the differential equation.

$$y' = y^2 \sin x$$

First, we'll write the equation in Leibniz notation. This makes it easier for us to separate the variables.

$$\frac{dy}{dx} = y^2 \sin x$$

Next, we'll separate the variables, collecting y's on the left and x's on the right.

$$dy = y^2 \sin x \ dx$$

$$\frac{dy}{y^2} = \sin x \ dx$$

$$\frac{1}{y^2} \ dy = \sin x \ dx$$

With variables separated, and integrating both sides, we get

$$\int \frac{1}{y^2} \ dy = \int \sin x \ dx$$

$$\int y^{-2} \ dy = \int \sin x \ dx$$

$$-y^{-1} = -\cos x + C$$

Note: You can leave out the constant of integration on the left side, because in future steps it would be absorbed into the constant on the right side.

$$-\frac{1}{y} = -\cos x + C$$

$$\frac{1}{y} = \cos x + C$$

Note: We just multiplied through both sides by -1, but we didn't change the sign on C, because the negative can always be absorbed into the constant.

$$1 = y(\cos x + C)$$

$$y = \frac{1}{\cos x + C}$$

Sometimes we'll encounter separable differential equations with initial conditions provided. Using the same method we used in the last example, we can find the general solution, and then plug in the initial condition(s) to find a particular solution to the differential equation.

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